3 Integration

3.1 Differential Forms

We will now consider integration in several variables. In order to smooth our discussion, we need to consider the concept of differential forms.

226 Definition Consider n variables

$$x_1, x_2, \ldots, x_n$$

in n-dimensional space (used as the names of the axes), and let

$$\mathbf{a}_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{nj} \end{bmatrix} \in \mathbb{R}^n, \quad 1 \leq j \leq k,$$

be $k \leq n$ vectors in $\mathbb{R}^n$. Moreover, let $\{j_1, j_2, \ldots, j_k\} \subseteq \{1, 2, \ldots, n\}$ be a collection of k sub-indices. An *elementary k-differential form ($k > 1$) acting on the vectors* $\mathbf{a}_j$, $1 \leq j \leq k$ is defined and denoted by

$$\mathrm{d}x_{j_1} \wedge \mathrm{d}x_{j_2} \wedge \cdots \wedge \mathrm{d}x_{j_k}(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k) = \det \begin{bmatrix} a_{j_1 1} & a_{j_1 2} & \cdots & a_{j_1 k} \\ a_{j_2 1} & a_{j_2 2} & \cdots & a_{j_2 k} \\ \vdots & \vdots & \cdots & \vdots \\ a_{j_k 1} & a_{j_k 2} & \cdots & a_{j_k k} \end{bmatrix}.$$

In other words, $\mathrm{d}x_{j_1} \wedge \mathrm{d}x_{j_2} \wedge \cdots \wedge \mathrm{d}x_{j_k}(\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k)$ is the $x_{j_1} x_{j_2} \ldots x_{j_k}$ component of the signed k-volume of a k-parallelotope in $\mathbb{R}^n$ spanned by $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k$.

☞ *By virtue of being a determinant, the wedge product $\wedge$ of differential forms has the following properties*

❶ **anti-commutativity:** $\mathrm{d}a \wedge \mathrm{d}b = -\mathrm{d}b \wedge \mathrm{d}a$.

❷ **linearity:** $\mathrm{d}(a + b) = \mathrm{d}a + \mathrm{d}b$.

❸ **scalar homogeneity:** *if* $\lambda \in \mathbb{R}$, *then* $\mathrm{d}\lambda a = \lambda \mathrm{d}a$.

❹ **associativity:** $(\mathrm{d}a \wedge \mathrm{d}b) \wedge \mathrm{d}c = \mathrm{d}a \wedge (\mathrm{d}b \wedge \mathrm{d}c)$.[1]

☞ *Anti-commutativity yields*

$$\mathrm{d}a \wedge \mathrm{d}a = 0.$$

227 Example Consider

$$\mathbf{a} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \in \mathbb{R}^3.$$

Then

$$\mathrm{d}x(\mathbf{a}) = \det(1) = 1,$$
$$\mathrm{d}y(\mathbf{a}) = \det(0) = 0,$$

[1] Notice that associativity does not hold for the wedge product of *vectors*.

$$\mathrm{d}z(\mathbf{a}) = \det(-1) = -1,$$

are the (signed) 1-volumes (that is, the length) of the projections of a onto the coordinate axes.

228 Example In $\mathbb{R}^3$ we have $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}x = 0$, since we have a repeated variable.

229 Example In $\mathbb{R}^3$ we have

$$\mathrm{d}x \wedge \mathrm{d}z + 5\mathrm{d}z \wedge \mathrm{d}x + 4\mathrm{d}x \wedge \mathrm{d}y - \mathrm{d}y \wedge \mathrm{d}x + 12\mathrm{d}x \wedge \mathrm{d}x = -4\mathrm{d}x \wedge \mathrm{d}z + 5\mathrm{d}x \wedge \mathrm{d}y.$$

☞ *In order to avoid redundancy we will make the convention that if a sum of two or more terms have the same differential form up to permutation of the variables, we will simplify the summands and express the other differential forms in terms of the one differential form whose indices appear in increasing order.*

230 Definition A 0-differential form in $\mathbb{R}^n$ is simply a differentiable function in $\mathbb{R}^n$.

231 Definition A *k-differential form field in* $\mathbb{R}^n$ is an expression of the form

$$\omega = \sum_{1 \leq j_1 \leq j_2 \leq \cdots \leq j_k \leq n} a_{j_1 j_2 \ldots j_k} \mathrm{d}x_{j_1} \wedge \mathrm{d}x_{j_2} \wedge \cdots \mathrm{d}x_{j_k},$$

where the $a_{j_1 j_2 \ldots j_k}$ are differentiable functions in $\mathbb{R}^n$.

232 Example

$$g(x, y, z, w) = x + y^2 + z^3 + w^4$$

is a 0-form in $\mathbb{R}^4$.

233 Example An example of a 1-form field in $\mathbb{R}^3$ is

$$\omega = x\mathrm{d}x + y^2\mathrm{d}y + xyz^3\mathrm{d}z.$$

234 Example An example of a 2-form field in $\mathbb{R}^3$ is

$$\omega = x^2\mathrm{d}x \wedge \mathrm{d}y + y^2\mathrm{d}y \wedge \mathrm{d}z + \mathrm{d}z \wedge \mathrm{d}x.$$

235 Example An example of a 3-form field in $\mathbb{R}^3$ is

$$\omega = (x + y + z)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.$$

We shew now how to multiply differential forms.

236 Example The product of the 1-form fields in $\mathbb{R}^3$

$$\omega_1 = y\mathrm{d}x + x\mathrm{d}y,$$

$$\omega_2 = -2x\mathrm{d}x + 2y\mathrm{d}y,$$

is

$$\omega_1 \wedge \omega_2 = (2x^2 + 2y^2)\mathrm{d}x \wedge \mathrm{d}y.$$

237 Definition Let $f(x_1, x_2, \ldots, x_n)$ be a 0-form in $\mathbb{R}^n$. The *exterior derivative* $\mathrm{d}f$ of f is

$$\mathrm{d}f = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\mathrm{d}x_i.$$

Furthermore, if

$$\omega = f(x_1, x_2, \ldots, x_n)\mathrm{d}x_{j_1} \wedge \mathrm{d}x_{j_2} \wedge \cdots \wedge \mathrm{d}x_{j_k}$$

is a k-form in $\mathbb{R}^n$, the *exterior derivative* $\mathrm{d}\omega$ of ω is the $(k+1)$-form

$$\mathrm{d}\omega = \mathrm{d}f(x_1, x_2, \ldots, x_n) \wedge \mathrm{d}x_{j_1} \wedge \mathrm{d}x_{j_2} \wedge \cdots \wedge \mathrm{d}x_{j_k}.$$

238 Example If in $\mathbb{R}^2$, $\omega = x^3y^4$, then

$$\mathrm{d}(x^3y^4) = 3x^2y^4\mathrm{d}x + 4x^3y^3\mathrm{d}y.$$

239 Example If in $\mathbb{R}^2$, $\omega = x^2y\mathrm{d}x + x^3y^4\mathrm{d}y$ then

$$\begin{aligned}\mathrm{d}\omega &= \mathrm{d}(x^2y\mathrm{d}x + x^3y^4\mathrm{d}y)\\ &= (2xy\mathrm{d}x + x^2\mathrm{d}y) \wedge \mathrm{d}x + (3x^2y^4\mathrm{d}x + 4x^3y^3\mathrm{d}y) \wedge \mathrm{d}y\\ &= x^2\mathrm{d}y \wedge \mathrm{d}x + 3x^2y^4\mathrm{d}x \wedge \mathrm{d}y\\ &= (3x^2y^4 - x^2)\mathrm{d}x \wedge \mathrm{d}y\end{aligned}$$

240 Example Consider the change of variables $x = u + v, y = uv$. Then

$$\mathrm{d}x = \mathrm{d}u + \mathrm{d}v,$$

$$\mathrm{d}y = v\mathrm{d}u + u\mathrm{d}v,$$

whence

$$\mathrm{d}x \wedge \mathrm{d}y = (u - v)\mathrm{d}u \wedge \mathrm{d}v.$$

241 Example Consider the transformation of coordinates xyz into uvw coordinates given by

$$u = x + y + z, \; v = \frac{z}{y+z}, \; w = \frac{y+z}{x+y+z}.$$

Then

$$\mathrm{d}u = \mathrm{d}x + \mathrm{d}y + \mathrm{d}z,$$

$$\mathrm{d}v = -\frac{z}{(y+z)^2}\mathrm{d}y + \frac{y}{(y+z)^2}\mathrm{d}z,$$

$$\mathrm{d}w = -\frac{y+z}{(x+y+z)^2}\mathrm{d}x + \frac{x}{(x+y+z)^2}\mathrm{d}y + \frac{x}{(x+y+z)^2}\mathrm{d}z.$$

Multiplication gives

$$\begin{aligned}\mathrm{d}u \wedge \mathrm{d}v \wedge \mathrm{d}w &= \left(-\frac{zx}{(y+z)^2(x+y+z)^2} - \frac{y(y+z)}{(y+z)^2(x+y+z)^2}\right.\\ &\qquad \left. + \frac{z(y+z)}{(y+z)^2(x+y+z)^2} - \frac{xy}{(y+z)^2(x+y+z)^2}\right)\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z\\ &= \frac{z^2 - y^2 - zx - xy}{(y+z)^2(x+y+z)^2}\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.\end{aligned}$$

3.2 Zero-Manifolds

242 Definition A 0-*dimensional oriented manifold of* $\mathbb{R}^n$ is simply a point $\mathrm{x} \in \mathbb{R}^n$, with a choice of the $+$ or $-$ sign. A general oriented 0-manifold is a union of oriented points.

243 Definition Let $M = +\{\mathrm{b}\} \cup -\{\mathrm{a}\}$ be an oriented 0-manifold, and let ω be a 0-form. Then

$$\int_M \omega = \omega(\mathrm{b}) - \omega(\mathrm{a}).$$

☞ *$-\mathrm{x}$ has opposite orientation to $+\mathrm{x}$ and*

$$\int_{-\mathrm{x}} \omega = -\int_{+\mathrm{x}} \omega.$$

244 Example Let $M = -\{(1,0,0)\} \cup +\{(1,2,3)\} \cup -\{(0,-2,0)\}$[2] be an oriented 0-manifold, and let $\omega = x + 2y + z^2$. Then

$$\int_M \omega = -\omega((1,0,0)) + \omega(1,2,3) - \omega(0,0,3) = -(1) + (14) - (-4) = 17.$$

3.3 One-Manifolds

245 Definition A 1-*dimensional oriented manifold of* $\mathbb{R}^n$ is simply an oriented smooth curve $\Gamma \in \mathbb{R}^n$, with a choice of a $+$ orientation if the curve traverses in the direction of increasing t, or with a choice of a $-$ sign if the curve traverses in the direction of decreasing t. A general oriented 1-manifold is a union of oriented curves.

☞ *The curve $-\Gamma$ has opposite orientation to Γ and*

$$\int_{-\Gamma} \omega = -\int_{\Gamma} \omega.$$

If $\overrightarrow{\mathbf{f}} : \mathbb{R}^2 \to \mathbb{R}^2$ and if $\mathrm{d}\overrightarrow{\mathbf{r}} = \begin{bmatrix} \mathrm{d}x \\ \mathrm{d}y \end{bmatrix}$, the classical way of writing this is

$$\int_{\Gamma} \overrightarrow{\mathbf{f}} \bullet \mathrm{d}\overrightarrow{\mathbf{r}}.$$

We now turn to the problem of integrating 1-forms.

246 Example Calculate

$$\int_{\Gamma} xy\mathrm{d}x + (x+y)\mathrm{d}y$$

where Γ is the parabola $y = x^2,\ x \in [-1;2]$ oriented in the positive direction.

Solution: ► *We parametrise the curve as $x = t, y = t^2$. Then*

$$xy\mathrm{d}x + (x+y)\mathrm{d}y = t^3\mathrm{d}t + (t+t^2)\mathrm{d}t^2 = (3t^3 + 2t^2)\mathrm{d}t,$$

whence

$$\begin{aligned} \int_{\Gamma} \omega &= \int_{-1}^{2} (3t^3 + 2t^2)\mathrm{d}t \\ &= \left[\frac{2}{3}t^3 + \frac{3}{4}t^4\right]_{-1}^{2} \\ &= \frac{69}{4}. \end{aligned}$$

What would happen if we had given the curve above a different parametrisation? First observe that the curve travels from $(-1,1)$ to $(2,4)$ on the parabola $y = x^2$. These conditions are met with the parametrisation $x = \sqrt{t} - 1, y = (\sqrt{t}-1)^2, t \in [0;9]$. Then

$$\begin{aligned} xy\mathrm{d}x + (x+y)\mathrm{d}y &= (\sqrt{t}-1)^3\mathrm{d}(\sqrt{t}-1) + ((\sqrt{t}-1) + (\sqrt{t}-1)^2)\mathrm{d}(\sqrt{t}-1)^2 \\ &= (3(\sqrt{t}-1)^3 + 2(\sqrt{t}-1)^2)\mathrm{d}(\sqrt{t}-1) \\ &= \frac{1}{2\sqrt{t}}(3(\sqrt{t}-1)^3 + 2(\sqrt{t}-1)^2)\mathrm{d}t, \end{aligned}$$

[2] Do not confuse, say, $-\{(1,0,0)\}$ with $-(1,0,0) = (-1,0,0)$. The first one means that the point $(1,0,0)$ is given negative orientation, the second means that $(-1,0,0)$ is the additive inverse of $(1,0,0)$.

whence

$$\begin{aligned}\int_\Gamma \omega &= \int_0^9 \frac{1}{2\sqrt{t}}(3(\sqrt{t}-1)^3+2(\sqrt{t}-1)^2)\mathrm{d}t\\ &= \left[\frac{3t^2}{4}-\frac{7t^{3/2}}{3}+\frac{5t}{2}-\sqrt{t}\right]_0^9\\ &= \frac{69}{4},\end{aligned}$$

as before.

To solve this problem using Maple you may use the code below.

```
> with(Student[VectorCalculus]):
> LineInt( VectorField( <x*y,x+y> ), Path( <t,t^2>, t=-1..2));
```

◀

☞ *It turns out that if two different parametrisations of the same curve have the same orientation, then their integrals are equal. Hence, we only need to worry about finding a suitable parametrisation.*

247 Example Calculate the line integral

$$\int_\Gamma y\sin x\mathrm{d}x + x\cos y\mathrm{d}y,$$

where Γ is the line segment from $(0,0)$ to $(1,1)$ in the positive direction.

Solution: ▶ *This line has equation $y=x$, so we choose the parametrisation $x=y=t$. The integral is thus*

$$\begin{aligned}\int_\Gamma y\sin x\mathrm{d}x + x\cos y\mathrm{d}y &= \int_0^1 (t\sin t + t\cos t)\mathrm{d}t\\ &= [t(\sin x-\cos t)]_0^1 - \int_0^1 (\sin t-\cos t)\mathrm{d}t\\ &= 2\sin 1 - 1,\end{aligned}$$

upon integrating by parts.

To solve this problem using Maple you may use the code below.

```
> with(Student[VectorCalculus]):
> LineInt( VectorField( <y*sin(x),x*cos(y)> ), Line(<0,0>,<1,1>));
```

◀

248 Example Calculate the path integral

$$\int_\Gamma \frac{x+y}{x^2+y^2}\ \mathrm{d}y + \frac{x-y}{x^2+y^2}\ \mathrm{d}x$$

around the closed square $\Gamma = ABCD$ with $A=(1,1)$, $B=(-1,1)$, $C=(-1,-1)$, and $D=(1,-1)$ in the direction $ABCDA$.

Solution: ▶ *On AB, $y=1, \mathrm{d}y=0$, on BC, $x=-1, \mathrm{d}x=0$, on CD, $y=-1, \mathrm{d}y=0$, and on DA, $x=1, \mathrm{d}x=0$. The integral is thus*

$$\begin{aligned}\int_\Gamma \omega &= \int_{AB}\omega + \int_{BC}\omega + \int_{CD}\omega + \int_{DA}\omega\\ &= \int_1^{-1}\frac{x-1}{x^2+1}\ \mathrm{d}x + \int_1^{-1}\frac{y-1}{y^2+1}\ \mathrm{d}y + \int_{-1}^1\frac{x+1}{x^2+1}\ \mathrm{d}x + \int_{-1}^1\frac{y+1}{y^2+1}\ \mathrm{d}y\\ &= 4\int_{-1}^1\frac{1}{x^2+1}\ \mathrm{d}x\\ &= 4\arctan x|_{-1}^1\\ &= 2\pi.\end{aligned}$$

To solve this problem using Maple you may use the code below.

```
> with(Student[VectorCalculus]):
> LineInt( VectorField( <(x+y)/(x^2+y^2),(x-y)/(x^2+y^2)> ),
> LineSegments(<1,1>,<-1,1>,<-1,-1>,<1,-1>,<1,1>));
```

◀

☞ *When the integral is along a closed path, like in the preceding example, it is customary to use the symbol* $\oint_{\Gamma}$ *rather than* $\int_{\Gamma}$*. The positive direction of integration is that sense that when traversing the path, the area enclosed by the curve is to the left of the curve.*

249 Example Calculate the path integral

$$\oint_{\Gamma} x^2 \mathrm{d}y + y^2 \mathrm{d}x,$$

where Γ is the ellipse $9x^2 + 4y^2 = 36$ traversed once in the positive sense.

Solution: ► *Parametrise the ellipse as $x = 2\cos t, y = 3\sin t, t \in [0; 2\pi]$. Observe that when traversing this closed curve, the area of the ellipse is on the left hand side of the path, so this parametrisation traverses the curve in the positive sense. We have*

$$\begin{aligned}\oint_{\Gamma} \omega &= \int_0^{2\pi} ((4\cos^2 t)(3\cos t) + (9\sin t)(-2\sin t))\mathrm{d}t \\ &= \int_0^{2\pi} (12\cos^3 t - 18\sin^3 t)\mathrm{d}t \\ &= 0.\end{aligned}$$

To solve this problem using Maple you may use the code below.

```
> with(Student[VectorCalculus]):
> LineInt( VectorField( <y^2,x^2> ),Ellipse(9*x^2 + 4*y^2 -36));
```

◀

250 Definition Let Γ be a smooth curve. The integral

$$\int_{\Gamma} f(\mathbf{x})||\mathrm{d}\mathbf{x}||$$

is called the *path integral of f along Γ*.

251 Example Find $\int_{\Gamma} x||\mathrm{d}\mathbf{x}||$ where Γ is the triangle starting at $A : (-1, -1)$ to $B : (2, -2)$, and ending in $C : (1, 2)$.

Solution: ► *The lines passing through the given points have equations $L_{AB} : y = \frac{-x-4}{3}$, and $L_{BC} : y = -4x + 6$. On L_{AB}*

$$x||\mathrm{d}\mathbf{x}|| = x\sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = x\sqrt{1 + \left(-\frac{1}{3}\right)^2}\mathrm{d}x = \frac{x\sqrt{10}\mathrm{d}x}{3},$$

and on L_{BC}

$$x||\mathrm{d}\mathbf{x}|| = x\sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = x(\sqrt{1 + (-4)^2})\mathrm{d}x = x\sqrt{17}\mathrm{d}x.$$

Hence

$$\begin{aligned}\int_{\Gamma} x||\mathrm{d}\mathbf{x}|| &= \int_{L_{AB}} x||\mathrm{d}\mathbf{x}|| + \int_{L_{BC}} x||\mathrm{d}\mathbf{x}|| \\ &= \int_{-1}^{2} \frac{x\sqrt{10}\mathrm{d}x}{3} + \int_{2}^{1} x\sqrt{17}\mathrm{d}x \\ &= \frac{\sqrt{10}}{2} - \frac{3\sqrt{17}}{2}.\end{aligned}$$

To solve this problem using Maple you may use the code below.

```
>  with(Student[VectorCalculus]):
>  PathInt( x, [x,y]=LineSegments( <-1,-1>, <2,-2>,<1,2> ) );
```

◀

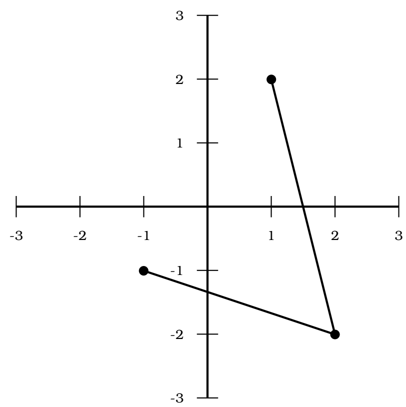

Figure 3.1: Example 251.

Homework

Problem 3.3.1 Consider $\int_C x\mathrm{d}x + y\mathrm{d}y$ and $\int_C xy||\mathrm{d}\mathbf{x}||$.

1. Evaluate $\int_C x\mathrm{d}x + y\mathrm{d}y$ where C is the straight line path that starts at $(-1,0)$ goes to $(0,1)$ and ends at $(1,0)$, by parametrising this path. Calculate also $\int_C xy||\mathrm{d}\mathbf{x}||$ using this parametrisation.
2. Evaluate $\int_C x\mathrm{d}x + y\mathrm{d}y$ where C is the semicircle that starts at $(-1,0)$ goes to $(0,1)$ and ends at $(1,0)$, by parametrising this path. Calculate also $\int_C xy||\mathrm{d}\mathbf{x}||$ using this parametrisation.

Problem 3.3.2 Find $\int_\Gamma x\mathrm{d}x + y\mathrm{d}y$ where Γ is the path shewn in figure 3.2, starting at $O(0,0)$ going on a straight line to $A\left(4\cos\frac{\pi}{6}, 4\sin\frac{\pi}{6}\right)$ and continuing on an arc of a circle to $B\left(4\cos\frac{\pi}{5}, 4\sin\frac{\pi}{5}\right)$.

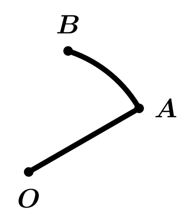

Figure 3.2: Problems 3.3.2 and 3.3.3.

Problem 3.3.3 Find $\int_\Gamma x||\mathrm{d}\mathbf{x}||$ where Γ is the path shewn in figure 3.2.

Problem 3.3.4 Find $\oint_\Gamma z\mathrm{d}x + x\mathrm{d}y + y\mathrm{d}z$ where Γ is the intersection of the sphere $x^2+y^2+z^2 = 1$ and the plane $x + y = 1$, traversed in the positive direction.

3.4 Closed and Exact Forms

252 Lemma (Poincaré Lemma) If ω is a p-differential form of continuously differentiable functions in $\mathbb{R}^n$ then
$$\mathrm{d}(\mathrm{d}\omega) = 0.$$

Proof: *We will prove this by induction on p. For $p = 0$ if*
$$\omega = f(x_1, x_2, \ldots, x_n)$$
then
$$\mathrm{d}\omega = \sum_{k=1}^{n} \frac{\partial f}{\partial x_k}\mathrm{d}x_k$$
and
$$\begin{aligned}\mathrm{d}(\mathrm{d}\omega) &= \sum_{k=1}^{n} \mathrm{d}\left(\frac{\partial f}{\partial x_k}\right) \wedge \mathrm{d}x_k \\ &= \sum_{k=1}^{n}\left(\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_k} \wedge \mathrm{d}x_j\right) \wedge \mathrm{d}x_k \\ &= \sum_{1\leq j\leq k\leq n}^{n} \left(\frac{\partial^2 f}{\partial x_j \partial x_k} - \frac{\partial^2 f}{\partial x_k \partial x_j}\right)\mathrm{d}x_j \wedge \mathrm{d}x_k \\ &= 0,\end{aligned}$$
since ω is continuously differentiable and so the mixed partial derivatives are equal. Consider now an arbitrary p-form, $p > 0$. Since such a form can be written as
$$\omega = \sum_{1\leq j_1\leq j_2\leq \cdots \leq j_p\leq n} a_{j_1 j_2 \ldots j_p}\mathrm{d}x_{j_1} \wedge \mathrm{d}x_{j_2} \wedge \cdots \mathrm{d}x_{j_p},$$
where the $a_{j_1 j_2 \ldots j_p}$ are continuous differentiable functions in $\mathbb{R}^n$, we have
$$\begin{aligned}\mathrm{d}\omega &= \sum_{1\leq j_1\leq j_2\leq \cdots \leq j_p\leq n} \mathrm{d}a_{j_1 j_2 \ldots j_p} \wedge \mathrm{d}x_{j_1} \wedge \mathrm{d}x_{j_2} \wedge \cdots \mathrm{d}x_{j_p} \\ &= \sum_{1\leq j_1\leq j_2\leq \cdots \leq j_p\leq n} \left(\sum_{i=1}^{n} \frac{\partial a_{j_1 j_2 \ldots j_p}}{\partial x_i}\mathrm{d}x_i\right) \wedge \mathrm{d}x_{j_1} \wedge \mathrm{d}x_{j_2} \wedge \cdots \mathrm{d}x_{j_p},\end{aligned}$$
it is enough to prove that for each summand
$$\mathrm{d}\left(\mathrm{d}a \wedge \mathrm{d}x_{j_1} \wedge \mathrm{d}x_{j_2} \wedge \cdots \mathrm{d}x_{j_p}\right) = 0.$$
But
$$\begin{aligned}\mathrm{d}\left(\mathrm{d}a \wedge \mathrm{d}x_{j_1} \wedge \mathrm{d}x_{j_2} \wedge \cdots \mathrm{d}x_{j_p}\right) &= \mathrm{d}\mathrm{d}a \wedge \left(\mathrm{d}x_{j_1} \wedge \mathrm{d}x_{j_2} \wedge \cdots \mathrm{d}x_{j_p}\right) \\ &\quad + \mathrm{d}a \wedge \mathrm{d}\left(\mathrm{d}x_{j_1} \wedge \mathrm{d}x_{j_2} \wedge \cdots \mathrm{d}x_{j_p}\right) \\ &= \mathrm{d}a \wedge \mathrm{d}\left(\mathrm{d}x_{j_1} \wedge \mathrm{d}x_{j_2} \wedge \cdots \mathrm{d}x_{j_p}\right),\end{aligned}$$
since $\mathrm{d}\mathrm{d}a = 0$ from the case $p = 0$. But an independent induction argument proves that
$$\mathrm{d}\left(\mathrm{d}x_{j_1} \wedge \mathrm{d}x_{j_2} \wedge \cdots \mathrm{d}x_{j_p}\right) = 0,$$
completing the proof. ❑

253 Definition A differential form ω is said to be *exact* if there is a continuously differentiable function F such that
$$\mathrm{d}F = \omega.$$

254 Example The differential form
$$x\mathrm{d}x + y\mathrm{d}y$$
is exact, since
$$x\mathrm{d}x + y\mathrm{d}y = \mathrm{d}\left(\frac{1}{2}(x^2 + y^2)\right).$$

255 Example The differential form
$$y\mathrm{d}x + x\mathrm{d}y$$
is exact, since
$$y\mathrm{d}x + x\mathrm{d}y = \mathrm{d}(xy).$$

256 Example The differential form
$$\frac{x}{x^2+y^2}\mathrm{d}x + \frac{y}{x^2+y^2}\mathrm{d}y$$
is exact, since
$$\frac{x}{x^2+y^2}\mathrm{d}x + \frac{y}{x^2+y^2}\mathrm{d}y = \mathrm{d}\left(\frac{1}{2}\log_e(x^2 + y^2)\right).$$

☞ *Let $\omega = \mathrm{d}F$ be an exact form. By the Poincaré Lemma Theorem 252, $\mathrm{d}\omega = \mathrm{dd}F = 0$. A result of Poincaré says that for certain domains (called* star-shaped domains*) the converse is also true, that is, if $\mathrm{d}\omega = 0$ on a star-shaped domain then ω is exact.*

257 Example Determine whether the differential form
$$\omega = \frac{2x(1-e^y)}{(1+x^2)^2}\mathrm{d}x + \frac{e^y}{1+x^2}\mathrm{d}y$$
is exact.

Solution: ► *Assume there is a function F such that*
$$\mathrm{d}F = \omega.$$
By the Chain Rule
$$\mathrm{d}F = \frac{\partial F}{\partial x}\mathrm{d}x + \frac{\partial F}{\partial y}\mathrm{d}y.$$
This demands that
$$\frac{\partial F}{\partial x} = \frac{2x(1-e^y)}{(1+x^2)^2},$$
$$\frac{\partial F}{\partial y} = \frac{e^y}{1+x^2}.$$
We have a choice here of integrating either the first, or the second expression. Since integrating the second expression (with respect to y) is easier, we find
$$F(x, y) = \frac{e^y}{1+x^2} + \phi(x),$$
where $\phi(x)$ is a function depending only on x. To find it, we differentiate the obtained expression for F with respect to x and find
$$\frac{\partial F}{\partial x} = -\frac{2xe^y}{(1+x^2)^2} + \phi'(x).$$

Comparing this with our first expression for $\frac{\partial F}{\partial x}$, *we find*

$$\phi'(x) = \frac{2x}{(1+x^2)^2},$$

that is

$$\phi(x) = -\frac{1}{1+x^2} + c,$$

where c *is a constant. We then take*

$$F(x,y) = \frac{e^y - 1}{1+x^2} + c.$$

◄

258 Example Is there a continuously differentiable function such that

$$\mathrm{d}F = \omega = y^2z^3\mathrm{d}x + 2xyz^3\mathrm{d}y + 3xy^2z^2\mathrm{d}z \quad ?$$

Solution: ► *We have*

$$\begin{aligned}
\mathrm{d}\omega &= (2yz^3\mathrm{d}y + 3y^2z^2\mathrm{d}z)\wedge \mathrm{d}x \\
&\quad +(2yz^3\mathrm{d}x + 2xz^3\mathrm{d}y + 6xyz^2\mathrm{d}z)\wedge \mathrm{d}y \\
&\quad +(3y^2z^2\mathrm{d}x + 6xyz^2\mathrm{d}y + 6xy^2z\mathrm{d}z)\wedge \mathrm{d}z \\
&= 0,
\end{aligned}$$

so this form is exact in a star-shaped domain. So put

$$\mathrm{d}F = \frac{\partial F}{\partial x}\mathrm{d}x + \frac{\partial F}{\partial y}\mathrm{d}y + \frac{\partial F}{\partial z}\mathrm{d}z = y^2z^3\mathrm{d}x + 2xyz^3\mathrm{d}y + 3xy^2z^2\mathrm{d}z.$$

Then

$$\begin{aligned}
\frac{\partial F}{\partial x} = y^2z^3 &\implies F = xy^2z^3 + a(y,z), \\
\frac{\partial F}{\partial y} = 2xyz^3 &\implies F = xy^2z^3 + b(x,z), \\
\frac{\partial F}{\partial z} = 3xy^2z^2 &\implies F = xy^2z^3 + c(x,y),
\end{aligned}$$

Comparing these three expressions for F, *we obtain* $F(x,y,z) = xy^2z^3$. ◄

We have the following equivalent of the Fundamental Theorem of Calculus.

259 Theorem Let $U \subseteq \mathbb{R}^n$ be an open set. Assume $\omega = \mathrm{d}F$ is an exact form, and Γ a path in U with starting point A and endpoint B. Then

$$\int_\Gamma \omega = \int_A^B \mathrm{d}F = F(B) - F(A).$$

In particular, if Γ is a simple closed path, then

$$\oint_\Gamma \omega = 0.$$

260 Example Evaluate the integral

$$\oint_\Gamma \frac{2x}{x^2+y^2}\mathrm{d}x + \frac{2y}{x^2+y^2}\,\mathrm{d}y$$

where Γ is the closed polygon with vertices at $A = (0,0)$, $B = (5,0)$, $C = (7,2)$, $D = (3,2)$, $E = (1,1)$, traversed in the order $ABCDEA$.

Solution: ▶ *Observe that*

$$\mathrm{d}\left(\frac{2x}{x^2+y^2}\mathrm{d}x+\frac{2y}{x^2+y^2}\,\mathrm{d}y\right)=-\frac{4xy}{(x^2+y^2)^2}\mathrm{d}y\wedge\mathrm{d}x-\frac{4xy}{(x^2+y^2)^2}\mathrm{d}x\wedge\mathrm{d}y=0,$$

and so the form is exact in a start-shaped domain. By virtue of Theorem 259, the integral is 0*.* ◀

261 Example Calculate the path integral

$$\oint_{\Gamma}(x^2-y)\mathrm{d}x+(y^2-x)\mathrm{d}y,$$

where Γ is a loop of $x^3+y^3-2xy=0$ traversed once in the positive sense.

Solution: ▶ *Since*

$$\frac{\partial}{\partial y}(x^2-y)=-1=\frac{\partial}{\partial x}(y^2-x),$$

the form is exact, and since this is a closed simple path, the integral is 0*.* ◀

3.5 Two-Manifolds

262 Definition A 2-*dimensional oriented manifold of* $\mathbb{R}^2$ is simply an open set (region) $D\in\mathbb{R}^2$, where the $+$ orientation is counter-clockwise and the $-$ orientation is clockwise. A general oriented 2-manifold is a union of open sets.

☞ *The region* $-D$ *has opposite orientation to* D *and*

$$\int_{-D}\omega=-\int_{D}\omega.$$

We will often write

$$\int_{D}f(x,y)\mathrm{d}A$$

where $\mathrm{d}A$ *denotes the* area element.

☞ *In this section, unless otherwise noticed, we will choose the positive orientation for the regions considered. This corresponds to using the area form* $\mathrm{d}x\mathrm{d}y$*.*

Let $D\subseteq\mathbb{R}^2$. Given a function $f:D\to\mathbb{R}$, the integral

$$\int\limits_{D}f\mathrm{d}A$$

is the sum of all the values of f restricted to D. In particular,

$$\int\limits_{D}\mathrm{d}A$$

is the area of D.

In order to evaluate double integrals, we need the following.

263 Theorem (Fubini's Theorem) Let $D=[a;b]\times[c;d]$, and let $f:A\to\mathbb{R}$ be continuous. Then

$$\int\limits_{D}f\mathrm{d}A=\int_a^b\left(\int_c^d f(x,y)\mathrm{d}y\right)\mathrm{d}x=\int_c^d\left(\int_a^b f(x,y)\mathrm{d}x\right)\mathrm{d}y$$

Fubini's Theorem allows us to convert the double integral into iterated (single) integrals.

264 Example

$$\begin{aligned}\int_{[0;1]\times[2;3]} xy\,\mathrm{d}A &= \int_0^1 \left(\int_2^3 xy\,\mathrm{d}y\right)\mathrm{d}x \\ &= \int_0^1 \left(\left[\frac{xy^2}{2}\right]_2^3\right)\mathrm{d}x \\ &= \int_0^1 \left(\frac{9x}{2} - 2x\right)\mathrm{d}x \\ &= \left[\frac{5x^2}{4}\right]_0^1 \\ &= \frac{5}{4}.\end{aligned}$$

Notice that if we had integrated first with respect to x we would have obtained the same result:

$$\begin{aligned}\int_2^3 \left(\int_0^1 xy\,\mathrm{d}x\right)\mathrm{d}y &= \int_2^3 \left(\left[\frac{x^2y}{2}\right]_0^1\right)\mathrm{d}y \\ &= \int_2^3 \left(\frac{y}{2}\right)\mathrm{d}x \\ &= \left[\frac{y^2}{4}\right]_2^3 \\ &= \frac{5}{4}.\end{aligned}$$

Also, this integral is "factorable into x and y pieces" meaning that

$$\begin{aligned}\int_{[0;1]\times[2;3]} xy\,\mathrm{d}A &= \left(\int_0^1 x\,\mathrm{d}x\right)\left(\int_2^3 y\,\mathrm{d}y\right) \\ &= \left(\frac{1}{2}\right)\left(\frac{5}{2}\right) \\ &= \frac{5}{4}\end{aligned}$$

To solve this problem using Maple you may use the code below.

```
> with(Student[VectorCalculus]):
> int(x*y, [x,y]=Region(0..1,2..3));
```

265 Example We have

$$\begin{aligned}\int_3^4\int_0^1 (x+2y)(2x+y)\ \mathrm{d}x\mathrm{d}y &= \int_3^4\int_0^1 (2x^2+5xy+2y^2)\ \mathrm{d}x\mathrm{d}y \\ &= \int_3^4 \left(\frac{2}{3}+\frac{5}{2}y+2y^2\right)\ \mathrm{d}y \\ &= \frac{409}{12}.\end{aligned}$$

To solve this problem using Maple you may use the code below.

```
> with(Student[VectorCalculus]):
> int((x + 2*y)*(2*x + y), [x,y]=Region(3..4,0..1));
```

In the cases when the domain of integration is not a rectangle, we decompose so that, one variable is kept constant.

266 Example Find $\int_D xy\ \mathrm{d}x\mathrm{d}y$ in the triangle with vertices $A:(-1,-1)$, $B:(2,-2)$, $C:(1,2)$.

Solution: ▶ *The lines passing through the given points have equations* $L_{AB} : y = \dfrac{-x-4}{3}$, $L_{BC} : y = -4x + 6$, $L_{CA} : y = \dfrac{3x+1}{2}$. *Now, we draw the region* carefully. *If we integrate first with respect to* y, *we must divide the region as in figure 3.3, because there are two upper lines which the upper value of* y *might be. The lower point of the dashed line is* $(1, -5/3)$. *The integral is thus*

$$\int_{-1}^{1} x \left(\int_{(-x-4)/3}^{(3x+1)/2} y \, \mathrm{d}y \right) \mathrm{d}x + \int_{1}^{2} x \left(\int_{(-x-4)/3}^{-4x+6} y \, \mathrm{d}y \right) \mathrm{d}x = -\frac{11}{8}.$$

If we integrate first with respect to x, *we must divide the region as in figure 3.4, because there are two left-most lines which the left value of* x *might be. The right point of the dashed line is* $(7/4, -1)$. *The integral is thus*

$$\int_{-2}^{-1} y \left(\int_{-4-3y}^{(6-y)/4} x \, \mathrm{d}x \right) \mathrm{d}y + \int_{-1}^{2} y \left(\int_{(2y-1)/3}^{(6-y)/4} x \, \mathrm{d}x \right) \mathrm{d}y = -\frac{11}{8}.$$

To solve this problem using Maple you may use the code below.

```
> with(Student[VectorCalculus]):
> int(x*y, [x,y]=Triangle(<-1,-1>,<2,-2>,<1,2>);
```

◀

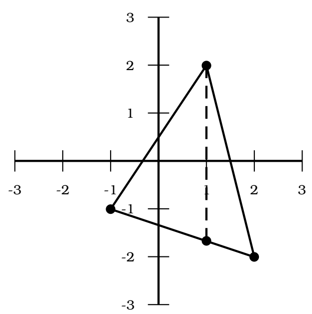

Figure 3.3: Example 266. Integration order $\mathrm{d}y\mathrm{d}x$.

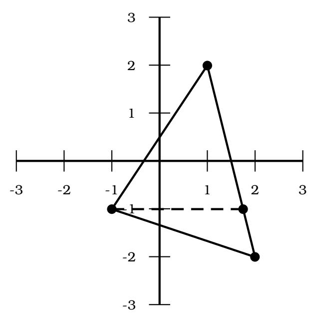

Figure 3.4: Example 266. Integration order $\mathrm{d}x\mathrm{d}y$.

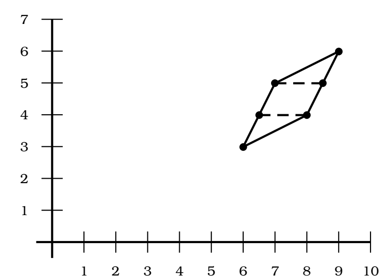

Figure 3.5: Example 267.

267 Example Consider the region inside the parallelogram P with vertices at $A : (6, 3)$, $B : (8, 4)$, $C : (9, 6)$, $D : (7, 5)$, as in figure 3.5. Find

$$\int_{P} xy \;\; \mathrm{d}x\mathrm{d}y.$$

Solution: ▶ *The lines joining the points have equations*

$$L_{AB} : \; y = \frac{x}{2},$$

$$L_{BC} : \; y = 2x - 12,$$

$$L_{CD} : \; y = \frac{x}{2} + \frac{3}{2},$$

$$L_{DA} : \; y = 2x - 9.$$

The integral is thus

$$\int_3^4 \int_{(y+9)/2}^{2y} xy \, dxdy + \int_4^5 \int_{(y+9)/2}^{(y+12)/2} xy \, dxdy + \int_5^6 \int_{2y-3}^{(y+12)/2} xy \, dxdy = \frac{409}{4}.$$

To solve this problem using Maple you may use the code below. Notice that we have split the parallelogram into two triangles.

```
> with(Student[VectorCalculus]):
> int(x*y, [x,y]=Triangle(<6,3>,<8,4>,<7,5>))
> + int(x*y, [x,y]=Triangle(<8,4>,<9,6>,<7,5>));
```

◀

268 Example Find

$$\int_D \frac{y}{x^2+1} dxdy$$

where

$$D = \{(x, y) \in \mathbb{R}^2 | x \geq 0, x^2 + y^2 \leq 1\}.$$

Solution: ► *The integral is* 0. *Observe that if* $(x, y) \in D$ *then* $(x, -y) \in D$. *Also,* $f(x, -y) = -f(x, y)$. ◀

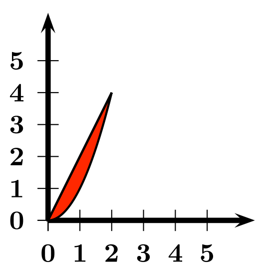

Figure 3.6: Example 269.

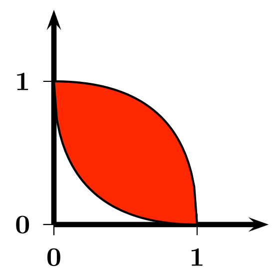

Figure 3.7: Example 270.

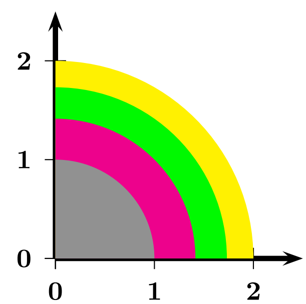

Figure 3.8: Example 271.

269 Example Find

$$\int_0^4 \left(\int_{y/2}^{\sqrt{y}} e^{y/x} \quad dx \right) dy.$$

Solution: ► *We have*

$$0 \leq y \leq 4, \quad \frac{y}{2} \leq x \leq \sqrt{y} \implies 0 \leq x \leq 2, \quad x^2 \leq y \leq 2x.$$

We then have

$$\begin{aligned} \int_0^4 \left(\int_{y/2}^{\sqrt{y}} e^{y/x} dx \right) dy &= \int_0^2 \left(\int_{x^2}^{2x} e^{y/x} dy \right) dx \\ &= \int_0^2 \left(x e^{y/x} \Big|_{x^2}^{2x} \right) dx \\ &= \int_0^2 (x e^2 - x e^x) \, dx \\ &= 2e^2 - (2e^2 - e^2 + 1) \\ &= e^2 - 1 \end{aligned}$$

◀

270 Example Find the area of the region

$$R = \{(x, y) \in \mathbb{R}^2 : \sqrt{x} + \sqrt{y} \geq 1, \quad \sqrt{1-x} + \sqrt{1-y} \geq 1\}.$$

Solution: ► *The area is given by*

$$\begin{aligned}\int_D \mathrm{d}A &= \int_0^1 \left(\int_{(1-\sqrt{x})^2}^{1-(1-\sqrt{1-x})^2} \mathrm{d}y\right) \mathrm{d}x \\ &= 2\int_0^1 (\sqrt{1-x} + \sqrt{x} - 1)\mathrm{d}x \\ &= \frac{2}{3}.\end{aligned}$$

◄

271 Example Evaluate $\int_R \llbracket x^2 + y^2 \rrbracket \mathrm{d}A$, where R is the rectangle $[0; \sqrt{2}] \times [0; \sqrt{2}]$.

Solution: ► *The function* $(x, y) \mapsto \llbracket x^2 + y^2 \rrbracket$ *jumps every time* $x^2 + y^2$ *is an integer. For* $(x, y) \in R$*, we have* $0 \leq x^2 + y^2 \leq (\sqrt{2})^2 + (\sqrt{2})^2 = 4$*. Thus we decompose* R *as the union of the*

$$R_k = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, k \leq x^2 + y^2 < k + 1\}, \qquad k \in \{1, 2, 3\}.$$

$$\begin{aligned}\int_R \llbracket x^2 + y^2 \rrbracket \mathrm{d}A &= \sum_{1 \leq k \leq 3} \int_{R_k} \llbracket x^2 + y^2 \rrbracket \mathrm{d}A \\ &= \iint\limits_{1 \leq x^2+y^2<2, x\geq 0, y \geq 0} 1\mathrm{d}A + \iint\limits_{2 \leq x^2+y^2<3, x\geq 0, y \geq 0} 2\mathrm{d}A + \iint\limits_{3 \leq x^2+y^2<4, x\geq 0, y \geq 0} 3\mathrm{d}A.\end{aligned}$$

Now the integrals can be computed by realising that they are areas of quarter annuli, and so,

$$\iint\limits_{k \leq x^2+y^2<k+1, x\geq 0, y \geq 0} k\mathrm{d}A = k \cdot \frac{1}{4} \cdot \pi(k + 1 - k) = \frac{\pi k}{4}.$$

Hence

$$\int_R \llbracket x^2 + y^2 \rrbracket \mathrm{d}A = \frac{\pi}{4}(1 + 2 + 3) = \frac{3\pi}{2}.$$

◄

Homework

Problem 3.5.1 Evaluate the iterated integral $\int_1^3 \int_0^x \frac{1}{x} \mathrm{d}y\mathrm{d}x$.

Problem 3.5.2 Let S be the interior and boundary of the triangle with vertices $(0, 0)$, $(2, 1)$, and $(2, 0)$. Find $\int\limits_S y\mathrm{d}A$.

Problem 3.5.3 Let

$$S = \{(x, y) \in \mathbb{R}^2 : x \geq 0, \ y \geq 0, \ 1 \leq x^2 + y^2 \leq 4\}.$$

Find $\int\limits_S x^2\mathrm{d}A$.

Problem 3.5.4 Find

$$\int\limits_D xy\mathrm{d}x\mathrm{d}y$$

where

$$D = \{(x, y) \in \mathbb{R}^2 | y \geq x^2, x \geq y^2\}.$$

Problem 3.5.5 Find

$$\int\limits_D (x + y)(\sin x)(\sin y)\mathrm{d}A$$

where $D = [0; \pi]^2$.

Problem 3.5.6 Find $\int_0^1 \int_0^1 \min(x^2, y^2)\mathrm{d}x\mathrm{d}y$.

Problem 3.5.7 Find $\int_D xy\mathrm{d}x\mathrm{d}y$ where

$D = \{(x, y) \in \mathbb{R}^2 : x > 0, y > 0, 9 < x^2+y^2 < 16, 1 < x^2-y^2 < 16\}$.

Problem 3.5.8 Evaluate $\int_{\mathcal{R}} x\mathrm{d}A$ where $\mathcal{R}$ is the (unoriented) circular segment in figure 3.9, which is created by the intersection of regions

$$\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 16\}$$

and

$$\left\{(x, y) \in \mathbb{R}^2 : y \geq -\frac{\sqrt{3}}{3}x + 4\right\}.$$

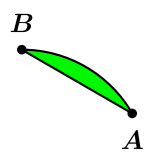

Figure 3.9: Problem 3.5.8.

Problem 3.5.9 Find $\int_0^1 \int_y^1 2e^{x^2}\mathrm{d}x\mathrm{d}y$

Problem 3.5.10 Evaluate $\int_{[0;1]^2} \min(x, y^2)\mathrm{d}A$.

Problem 3.5.11 Find $\int_{\mathcal{R}} xy\mathrm{d}A$, where $\mathcal{R}$ is the (unoriented) $\triangle OAB$ in figure 3.10 with $O(0, 0)$, $A(3, 1)$, and $B(4, 4)$.

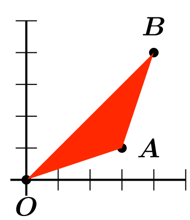

Figure 3.10: Problem 3.5.11.

Problem 3.5.12 Find

$$\int_D \log_e(1 + x + y)\mathrm{d}A$$

where

$$D = \{(x, y) \in \mathbb{R}^2 | x \geq 0, y \geq 0, x + y \leq 1\}.$$

Problem 3.5.13 Evaluate $\int_{[0;2]^2} \lfloor x + y^2 \rfloor \mathrm{d}A$.

Problem 3.5.14 Evaluate $\int_R \lfloor x+y \rfloor \mathrm{d}A$, where R is the rectangle $[0\,;1] \times [0\,;2]$.

Problem 3.5.15 Evaluate $\int_R x\mathrm{d}A$ where R is the quarter annulus in figure 3.11, which formed by the the area between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$ in the first quadrant.

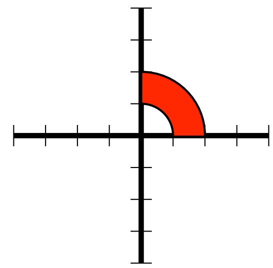

Figure 3.11: Problem 3.5.15.

Problem 3.5.16 Evaluate $\int_R x\mathrm{d}A$ where R is the E-shaped figure in figure 3.12.

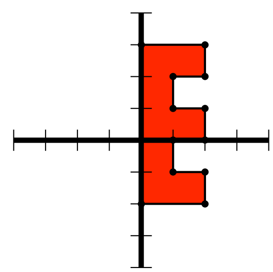

Figure 3.12: Problem 3.5.16.

Problem 3.5.17 Evaluate $\int_0^{\pi/2} \int_x^{\pi/2} \frac{\cos y}{y}\mathrm{d}y\mathrm{d}x$.

Problem 3.5.18 Find

$$\int_1^2 \left(\int_{\sqrt{x}}^x \sin\frac{\pi x}{2y}\,\mathrm{d}y\right)\,\mathrm{d}x + \int_2^4 \left(\int_{\sqrt{x}}^2 \sin\frac{\pi x}{2y}\,\mathrm{d}y\right)\,\mathrm{d}x.$$

Problem 3.5.19 Find

$$\int_D 2x(x^2 + y^2)\mathrm{d}A$$

where

$$D = \{(x, y) \in \mathbb{R}^2 : x^4 + y^4 + x^2 - y^2 \leq 1\}.$$

Problem 3.5.20 Find the area bounded by the ellipses $x^2 + \frac{y^2}{4} = 1$ and $\frac{x^2}{4} + y^2 = 1$, as in figure 3.13.

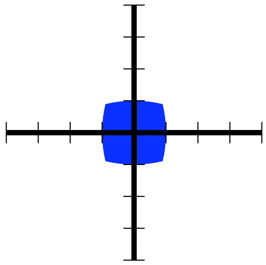

Figure 3.13: Problem 3.5.20.

Problem 3.5.21 Find

$$\int_D xy\mathrm{d}A$$

where

$$D = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, xy + y + x \leq 1\}.$$

Problem 3.5.22 Find

$$\int_D \log_e(1 + x^2 + y)\mathrm{d}A$$

where

$$D = \{(x, y) \in \mathbb{R}^2 : x \geq 0, y \geq 0, x^2 + y \leq 1\}.$$

Problem 3.5.23 Evaluate $\int_R x\mathrm{d}A$, where R is the region between the circles $x^2 + y^2 = 4$ and $x^2 + y^2 = 2y$, as shewn in figure 3.14.

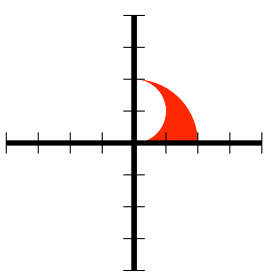

Figure 3.14: Problem 3.5.23.

Problem 3.5.24 Evaluate $\int_0^1 \int_{\sqrt{x}}^1 \frac{e^{x/y}}{y}\mathrm{d}y\mathrm{d}x$.

Problem 3.5.25 Find

$$\int_D |x - y|\mathrm{d}A$$

where

$$D = \{(x, y) \in \mathbb{R}^2 : |x| \leq 1, |y| \leq 1\}.$$

Problem 3.5.26 Find $\int_D (2x + 3y + 1)\ \mathrm{d}A$, where D is the triangle with vertices at $A(-1, -1)$, $B(2, -4)$, and $C(1, 3)$.

Problem 3.5.27 Let $f : [0; 1] \to]0; +\infty]$ be a decreasing function. Prove that

$$\frac{\int_0^1 xf^2(x)\mathrm{d}x}{\int_0^1 xf(x)\mathrm{d}x} \leq \frac{\int_0^1 f^2(x)\mathrm{d}x}{\int_0^1 f(x)\mathrm{d}x}.$$

Problem 3.5.28 Find

$$\int_D (xy(x + y))\mathrm{d}A$$

where

$$D = \{(x, y) \in \mathbb{R}^2 | x \geq 0, y \geq 0, x + y \leq 1\}.$$

Problem 3.5.29 Let $f, g : [0; 1] \to [0; 1]$ be continuous, with f increasing. Prove that

$$\int_0^1 (f \circ g)(x)\mathrm{d}x \leq \int_0^1 f(x)\mathrm{d}x + \int_0^1 g(x)\mathrm{d}x.$$

Problem 3.5.30 Compute $\int_S (xy + y^2)\mathrm{d}A$ where

$$S = \{(x, y) \in \mathbb{R}^2 : |x|^{1/2} + |y|^{1/2} \leq 1\}.$$

Problem 3.5.31 Evaluate

$$\int_0^a \int_0^b e^{\max(b^2x^2, a^2y^2)}\ \mathrm{d}y\mathrm{d}x,$$

where a and b are positive.

Problem 3.5.32 Find $\int_D \sqrt{xy}\ \mathrm{d}A$, where

$$D = \{(x, y) \in \mathbb{R}^2 : y \geq 0, (x + y)^2 \leq 2x\}.$$

Problem 3.5.33 A rectangle R on the plane is the disjoint union $R = \cup_{k=1}^N R_k$ of rectangles R_k. It is known that at least one side of each of the rectangles R_k is an integer. Shew that at least one side of R is an integer.

Problem 3.5.34 Evaluate
$$\int_0^1\int_0^1\cdots\int_0^1(x_1x_2\cdots x_n)\mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_n.$$

Problem 3.5.35 Evaluate
$$\int_0^1\int_0^1\cdots\int_0^1(x_1+x_2+\cdots+x_n)\,\mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_n.$$

Problem 3.5.36 Let I be the rectangle $[1;2]\times[1;2]$ and let f,g be continuous functions $f,g:[1;2]\to[1;2]$ such that $f(x)\leq g(x)$. Demonstrate that
$$\int_I (g(y)-f(x))\,\mathrm{d}x\mathrm{d}y\geq 0.$$

Problem 3.5.37 Find $\displaystyle\int_0^1\int_0^1 x^y\mathrm{d}x\mathrm{d}y$. Then demonstrate that $\displaystyle\int_0^1\frac{x-1}{\log x}\mathrm{d}x=\log 2$.

Problem 3.5.38 Evaluate
$$\int_0^4\int_0^{\sqrt{4-y}}\sqrt{12x-x^3}\mathrm{d}x\mathrm{d}y.$$

Problem 3.5.39 Evaluate $\displaystyle\int_0^2\int_y^2 y\sqrt{1+x^3}\mathrm{d}x\mathrm{d}y$.

Problem 3.5.40 Evaluate $\displaystyle\int_0^1\int_y^1\frac{xy}{\sqrt{1+x^4}}\mathrm{d}x\mathrm{d}y$.

Problem 3.5.41 Find
$$\int_D\frac{1}{(x+y)^4}\mathrm{d}A$$
where
$$D=\{(x,y)\in\mathbb{R}^2|x\geq 1,y\geq 1,x+y\leq 4\}.$$

Problem 3.5.42 Prove that
$$\int_0^{+\infty}\int_{2x}^{+\infty}\frac{xe^{-y}\sin y}{y^2}\mathrm{d}y\mathrm{d}x=\frac{1}{16}.$$

Problem 3.5.43 Prove that
$$\int_0^1\int_{y^2}^{y}\frac{y}{x\sqrt{x^2+y^2}}\mathrm{d}x\mathrm{d}y=\log(1+\sqrt{2}).$$

Problem 3.5.44 Prove that
$$\int_0^1\int_0^1\frac{x-y}{(x+y)^3}\mathrm{d}y\mathrm{d}x=\frac{1}{2}=-\int_0^1\int_0^1\frac{x-y}{(x+y)^3}\mathrm{d}x\mathrm{d}y.$$
Is this a contradiction to Fubini's Theorem?

Problem 3.5.45 Find
$$\int_D x\mathrm{d}A$$
where
$$D=\{(x,y)\in\mathbb{R}^2|y\geq 0,x-y+1\geq 0,x+2y-4\leq 0\}.$$

Problem 3.5.46 Evaluate
$$\lim_{n\to+\infty}\int_0^1\int_0^1\cdots\int_0^1\cos^2\left(\frac{\pi}{2n}(x_1+x_2+\cdots+x_n)\right)\mathrm{d}x_1\,\mathrm{d}x_2\ldots\,\mathrm{d}x$$

Problem 3.5.47 Let f,g be continuous functions in the interval $[a;b]$. Prove that
$$\frac{1}{2}\int_a^b\left(\int_a^b\det\begin{bmatrix}f(x) & g(x)\\ f(y) & g(y)\end{bmatrix}^2\mathrm{d}x\right)\mathrm{d}y$$
equals
$$\left(\int_a^b(f(x))^2\mathrm{d}x\right)\left(\int_a^b(g(x))^2\mathrm{d}x\right)-\left(\int_a^b(f(x)g(x))\mathrm{d}x\right)^2.$$
This is an integral analogue of Lagrange's Identity. Deduce Cauchy's Inequality for integrals,
$$\left(\int_a^b(f(x)g(x))\mathrm{d}x\right)^2\leq\left(\int_a^b(f(x))^2\mathrm{d}x\right)\left(\int_a^b(g(x))^2\mathrm{d}x\right).$$

Problem 3.5.48 Let $a\in\mathbb{R},n\in\mathbb{N}$, $a>0$, $n>0$. Let $f:[0;a]\to\mathbb{R}$ be continuous. Prove that
$$\int_0^a\int_0^{x_1}\cdots\int_0^{x_{n-2}}\int_0^{x_{n-1}}(f(x_1)f(x_2)\cdots f(x_n))\mathrm{d}x_n\mathrm{d}x_{n-1}\ldots\mathrm{d}x_2\mathrm{d}$$
equals
$$\frac{1}{n!}\left(\int_0^a f(x)\mathrm{d}x\right)^n.$$

3.6 Change of Variables

We now perform a multidimensional analogue of the change of variables theorem in one variable.

272 Theorem Let $(D,\Delta)\in(\mathbb{R}^n)^2$ be open, bounded sets in $\mathbb{R}^n$ with volume and let $g:\Delta\to D$ be a continuously differentiable bijective mapping such that $\det g'(u)\neq 0$, and both $|\det g'(u)|$, $\dfrac{1}{|\det g'(u)|}$

are bounded on Δ. For $f : D \to \mathbb{R}$ bounded and integrable, $f \circ g|\det g'(u)|$ is integrable on Δ and

$$\int \cdots \int_D f = \int \cdots \int_\Delta (f \circ g)|\det g'(u)|,$$

that is

$$\int \cdots \int_D f(x_1, x_2, \ldots, x_n)\mathrm{d}x_1 \wedge \mathrm{d}x_2 \wedge \ldots \wedge \mathrm{d}x_n$$
$$= \int \cdots \int_\Delta f(g(u_1, u_2, \ldots, u_n))|\det g'(u)|\mathrm{d}u_1 \wedge \mathrm{d}u_2 \wedge \ldots \wedge \mathrm{d}u_n.$$

One normally chooses changes of variables that map into rectangular regions, or that simplify the integrand. Let us start with a rather trivial example.

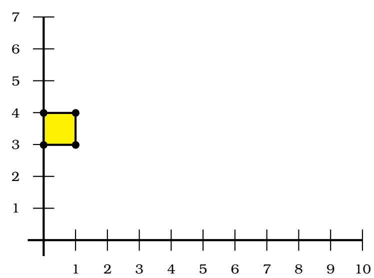

Figure 3.15: Example 273. xy-plane.

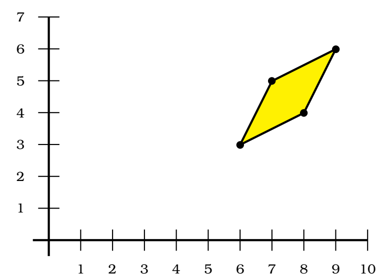

Figure 3.16: Example 273. uv-plane.

273 Example Evaluate the integral

$$\int_3^4 \int_0^1 (x + 2y)(2x + y)\mathrm{d}x\mathrm{d}y.$$

Solution: ► *Observe that we have already computed this integral in example 265. Put*

$$u = x + 2y \implies \mathrm{d}u = \mathrm{d}x + 2\mathrm{d}y,$$

$$v = 2x + y \implies \mathrm{d}v = 2\mathrm{d}x + \mathrm{d}y,$$

giving

$$\mathrm{d}u \wedge \mathrm{d}v = -3\mathrm{d}x \wedge \mathrm{d}y.$$

Now,

$$(u, v) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

is a linear transformation, and hence it maps quadrilaterals into quadrilaterals. The corners of the rectangle in the area of integration in the xy-plane are $(0, 3)$, $(1, 3)$, $(1, 4)$, and $(0, 4)$, (traversed counter-clockwise) and they map into $(6, 3)$, $(7, 5)$, $(9, 6)$, and $(8, 4)$, respectively, in the uv-plane (see figure 3.16). The form $\mathrm{d}x \wedge \mathrm{d}y$ has opposite orientation to $\mathrm{d}u \wedge \mathrm{d}v$ so we use

$$\mathrm{d}v \wedge \mathrm{d}u = 3\mathrm{d}x \wedge \mathrm{d}y$$

instead. The integral sought is

$$\frac{1}{3}\int_P uv\,\mathrm{d}v\mathrm{d}u = \frac{409}{12},$$

from example 267. ◀

274 Example The integral

$$\int_{[0;1]^2} (x^4 - y^4)\mathrm{d}A = \int_0^1 \left(\frac{1}{5} - y^4\right)\mathrm{d}y = 0.$$

Evaluate it using the change of variables $u = x^2 - y^2, v = 2xy$.

Solution: ▶ *First we find*

$$\mathrm{d}u = 2x\mathrm{d}x - 2y\mathrm{d}y,$$

$$\mathrm{d}v = 2y\mathrm{d}x + 2x\mathrm{d}y,$$

and so

$$\mathrm{d}u \wedge \mathrm{d}v = (4x^2 + 4y^2)\mathrm{d}x \wedge \mathrm{d}y.$$

We now determine the region Δ *into which the square* $D = [0;1]^2$ *is mapped. We use the fact that boundaries will be mapped into boundaries. Put*

$$AB = \{(x, 0) : 0 \le x \le 1\},$$

$$BC = \{(1, y) : 0 \le y \le 1\},$$

$$CD = \{(1 - x, 1) : 0 \le x \le 1\},$$

$$DA = \{(0, 1 - y) : 0 \le y \le 1\}.$$

On AB *we have* $u = x, v = 0$. *Since* $0 \le x \le 1$, AB *is thus mapped into the line segment* $0 \le u \le 1, v = 0$.

On BC *we have* $u = 1 - y^2, v = 2y$. *Thus* $u = 1 - \frac{v^2}{4}$. *Hence* BC *is mapped to the portion of the parabola* $u = 1 - \frac{v^2}{4}, 0 \le v \le 2$.

On CD *we have* $u = (1 - x)^2 - 1, v = 2(1 - x)$. *This means that* $u = \frac{v^2}{4} - 1, 0 \le v \le 2$.

Finally, on DA, *we have* $u = -(1 - y)^2, v = 0$. *Since* $0 \le y \le 1$, DA *is mapped into the line segment* $-1 \le u \le 0, v = 0$. *The region* Δ *is thus the area in the* uv *plane enclosed by the parabolas* $u \le \frac{v^2}{4} - 1, u \le 1 - \frac{v^2}{4}$ *with* $-1 \le u \le 1, 0 \le v \le 2$.

We deduce that

$$\begin{aligned}
\int_{[0;1]^2} (x^4 - y^4)\mathrm{d}A &= \int_\Delta (x^4 - y^4)\frac{1}{4(x^2 + y^2)}\mathrm{d}u\mathrm{d}v \\
&= \frac{1}{4}\int_\Delta (x^2 - y^2)\mathrm{d}u\mathrm{d}v \\
&= \frac{1}{4}\int_\Delta u\mathrm{d}u\mathrm{d}v \\
&= \frac{1}{4}\int_0^2 \left(\int_{v^2/4-1}^{1-v^2/4} u\mathrm{d}u\right)\mathrm{d}v \\
&= 0,
\end{aligned}$$

as before. ◀

275 Example Find

$$\int_D e^{(x^3+y^3)/xy} \quad \mathrm{d}A$$

where

$$D = \{(x,y) \in \mathbb{R}^2 | y^2 - 2px \le 0, x^2 - 2py \le 0, p \in]0;+\infty[\text{ fixed}\},$$

using the change of variables $x = u^2v,\ y = uv^2$.

Solution: ► *We have*

$$\begin{aligned} \mathrm{d}x &= 2uv\mathrm{d}u + u^2\mathrm{d}v, \\ \mathrm{d}y &= v^2\mathrm{d}u + 2uv\mathrm{d}v, \\ \mathrm{d}x \wedge \mathrm{d}y &= 3u^2v^2\mathrm{d}u \wedge \mathrm{d}v. \end{aligned}$$

The region transforms into

$$\Delta = \{(u,v) \in \mathbb{R}^2 | 0 \le u \le (2p)^{1/3},\ 0 \le v \le (2p)^{1/3}\}.$$

The integral becomes

$$\begin{aligned} \int_D f(x,y)\mathrm{d}x\mathrm{d}y &= \int_\Delta \exp\left(\frac{u^6v^3 + u^3v^6}{u^3v^3}\right)(3u^2v^2)\ \mathrm{d}u\mathrm{d}v \\ &= 3\int_\Delta e^{u^3}e^{v^3}u^2v^2\ \mathrm{d}u\mathrm{d}v \\ &= \frac{1}{3}\left(\int_0^{(2p)^{1/3}} 3u^2e^{u^3}\ \mathrm{d}u\right)^2 \\ &= \frac{1}{3}(e^{2p}-1)^2. \end{aligned}$$

As an exercise, you may try the (more natural) substitution $x^3 = u^2v, y^3 = v^2u$ and verify that the same result is obtained. ◄

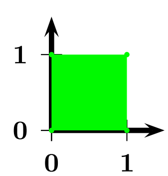

Figure 3.17: Example 276. xy-plane.

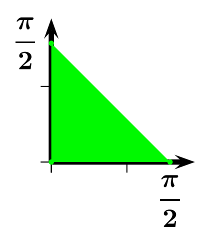

Figure 3.18: Example 276. uv-plane.

276 Example In this problem we will follow an argument of Calabi, Beukers, and Kock to prove that $\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

1. Prove that if $S = \sum_{n=1}^{+\infty} \frac{1}{n^2}$, then $\frac{3}{4}S = \sum_{n=1}^{+\infty} \frac{1}{(2n-1)^2}$.

2. Prove that $\sum_{n=1}^{+\infty} \frac{1}{(2n-1)^2} = \int_0^1 \int_0^1 \frac{\mathrm{d}x\mathrm{d}y}{1-x^2y^2}$.

3. Use the change of variables $x = \dfrac{\sin u}{\cos v}$, $y = \dfrac{\sin v}{\cos u}$ in order to evaluate $\displaystyle\int_0^1 \int_0^1 \frac{\mathrm{d}x\mathrm{d}y}{1 - x^2y^2}$.

Solution: ▶

1. *Observe that the sum of the even terms is*

$$\sum_{n=1}^{+\infty} \frac{1}{(2n)^2} = \frac{1}{4}\sum_{n=1}^{+\infty} \frac{1}{n^2} = \frac{1}{4}S,$$

a quarter of the sum, hence the sum of the odd terms must be three quarters of the sum, $\dfrac{3}{4}S$.

2. *Observe that*

$$\frac{1}{2n-1} = \int_0^1 x^{2n-2}\mathrm{d}x \implies \left(\frac{1}{2n-1}\right)^2 = \left(\int_0^1 x^{2n-2}\mathrm{d}x\right)\left(\int_0^1 y^{2n-2}\mathrm{d}y\right) = \int_0^1\int_0^1 (xy)^{2n-2}\mathrm{d}x\mathrm{d}y.$$

Thus

$$\sum_{n=1}^{+\infty} \frac{1}{(2n-1)^2} = \sum_{n=1}^{+\infty} \int_0^1\int_0^1 (xy)^{2n-2}\mathrm{d}x\mathrm{d}y = \int_0^1\int_0^1 \sum_{n=1}^{+\infty} (xy)^{2n-2}\mathrm{d}x\mathrm{d}y = \int_0^1\int_0^1 \frac{\mathrm{d}x\mathrm{d}y}{1-x^2y^2},$$

as claimed.[3]

3. *If* $x = \dfrac{\sin u}{\cos v}$, $y = \dfrac{\sin v}{\cos u}$, *then*

$$\mathrm{d}x = (\cos u)(\sec v)\mathrm{d}u + (\sin u)(\sec v)(\tan v)\mathrm{d}v, \quad \mathrm{d}y = (\sec u)(\tan u)(\sin v)\mathrm{d}u + (\sec u)(\cos v)\mathrm{d}v,$$

from where

$$\mathrm{d}x \wedge \mathrm{d}y = \mathrm{d}u \wedge \mathrm{d}v - (\tan^2 u)(\tan^2 v)\mathrm{d}u \wedge \mathrm{d}v = \left(1 - (\tan^2 u)(\tan^2 v)\right)\mathrm{d}u \wedge \mathrm{d}v.$$

Also,

$$1 - x^2y^2 = 1 - \frac{\sin^2 v}{\cos^2 v}\cdot\frac{\sin^2 v}{\cos^2 u} = 1 - (\tan^2 u)(\tan^2 v).$$

This gives

$$\frac{\mathrm{d}x\mathrm{d}y}{1-x^2y^2} = \mathrm{d}u\mathrm{d}v.$$

We now have to determine the region that the transformation $x = \dfrac{\sin u}{\cos v}$, $y = \dfrac{\sin v}{\cos u}$ *forms in the* uv*-plane. Observe that*

$$u = \arctan x\sqrt{\frac{1-y^2}{1-x^2}}, \qquad v = \arctan y\sqrt{\frac{1-x^2}{1-y^2}}.$$

This means that the square in the xy*-plane in figure 3.17 is transformed into the triangle in the* uv*-plane in figure 3.18.*
We deduce,

$$\int_0^1\int_0^1 \frac{\mathrm{d}x\mathrm{d}y}{1-x^2y^2} = \int_0^{\pi/2}\int_0^{\pi/2-v} \mathrm{d}u\mathrm{d}v = \int_0^{\pi/2} (\pi/2 - v)\,\mathrm{d}v = \left(\frac{\pi}{2}v - \frac{v^2}{2}\right)\Big|_0^{\pi/2} = \frac{\pi^2}{4} - \frac{\pi^2}{8} = \frac{\pi^2}{8}.$$

Finally,

$$\frac{3}{4}S = \frac{\pi^2}{8} \implies S = \frac{\pi^2}{6}.$$

◀

[3] This exchange of integral and sum needs justification. We will accept it for our purposes.

Homework

Problem 3.6.1 Let $D' = \{(u, v) \in \mathbb{R}^2 : u \leq 1, -u \leq v \leq u\}$. Consider

$$\Phi : \begin{array}{ccc} \mathbb{R}^2 & \to & \mathbb{R}^2 \\ (u, v) & \mapsto & \left(\dfrac{u+v}{2}, \dfrac{u-v}{2}\right) \end{array}.$$

❶ Find the image of Φ on D', that is, find $D = \Phi(D')$.

❷ Find

$$\int_D (x+y)^2 e^{x^2-y^2} \mathrm{d}A.$$

Problem 3.6.2 Using the change of variables $x = u^2 - v^2$, $y = 2uv$, $u \geq 0$, $v \geq 0$, evaluate $\int_R \sqrt{x^2+y^2}\mathrm{d}A$, where

$$R = \{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1, 0 \leq y \leq 2\sqrt{1-|x|}\}$$

Problem 3.6.3 Using the change of variables $u = x - y$ and $v = x + y$, evaluate $\int_R \dfrac{x-y}{x+y}\mathrm{d}A$, where R is the square with vertices at $(0, 2)$, $(1, 1)$, $(2, 2)$, $(1, 3)$.

Problem 3.6.4 Find $\int\limits_D f(x, y)\mathrm{d}A$ where

$$D = \{(x, y) \in \mathbb{R}^2 | a \leq xy \leq b, y \geq x \geq 0, y^2 - x^2 \leq 1, (a, b) \in \mathbb{R}^2, 0 < a < b\}$$

and $f(x, y) = y^4 - x^4$ by using the change of variables $u = xy, v = y^2 - x^2$.

Problem 3.6.5 Use the following steps (due to Tom Apostol) in order to prove that

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

❶ Use the series expansion

$$\frac{1}{1-t} = 1 + t + t^2 + t^3 + \cdots \quad |t| < 1,$$

in order to prove (formally) that

$$\int_0^1 \int_0^1 \frac{\mathrm{d}x\mathrm{d}y}{1-xy} = \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

❷ Use the change of variables $u = x + y, v = x - y$ to shew that

$$\int_0^1 \int_0^1 \frac{\mathrm{d}x\mathrm{d}y}{1-xy} = 2\int_0^1 \left(\int_{-u}^{u} \frac{\mathrm{d}v}{4-u^2+v^2}\right) \mathrm{d}u + 2\int_1^2 \left(\int_{u-2}^{2-u} \frac{\mathrm{d}v}{4-u^2+v^2}\right) \mathrm{d}u.$$

❸ Shew that the above integral reduces to

$$2\int_0^1 \frac{2}{\sqrt{4-u^2}} \arctan \frac{u}{\sqrt{4-u^2}} \mathrm{d}u + 2\int_1^2 \frac{2}{\sqrt{4-u^2}} \arctan \frac{2-u}{\sqrt{4-u^2}} \mathrm{d}u.$$

❹ Finally, prove that the above integral is $\dfrac{\pi^2}{6}$ by using the substitution $\theta = \arcsin \dfrac{u}{2}$.

3.7 Change to Polar Coordinates

One of the most common changes of variable is the passage to polar coordinates where

$$x = \rho\cos\theta \implies \mathrm{d}x = \cos\theta\mathrm{d}\rho - \rho\sin\theta\mathrm{d}\theta,$$
$$y = \rho\sin\theta \implies \mathrm{d}y = \sin\theta\mathrm{d}\rho + \rho\cos\theta\mathrm{d}\theta,$$

whence

$$\mathrm{d}x \wedge \mathrm{d}y = (\rho\cos^2\theta + \rho\sin^2\theta)\mathrm{d}\rho \wedge \mathrm{d}\theta = \rho\mathrm{d}\rho \wedge \mathrm{d}\theta.$$

277 Example Find

$$\int_D xy\sqrt{x^2+y^2}\mathrm{d}A$$

where

$$D = \{(x,y) \in \mathbb{R}^2 | x \geq 0, y \geq 0, y \leq x, x^2+y^2 \leq 1\}.$$

Solution: ▸ *We use polar coordinates. The region D transforms into the region*

$$\Delta = [0;1] \times \left[0;\frac{\pi}{4}\right].$$

Therefore the integral becomes

$$\begin{aligned}\int_\Delta \rho^4\cos\theta\sin\theta\,\mathrm{d}\rho\mathrm{d}\theta &= \left(\int_0^{\pi/4}\cos\theta\sin\theta\,\mathrm{d}\theta\right)\left(\int_0^1\rho^4\,\mathrm{d}\rho\right)\\ &= \frac{1}{20}.\end{aligned}$$

◂

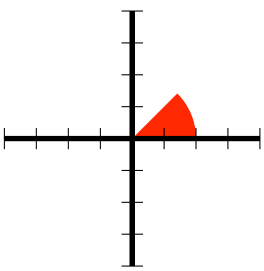

Figure 3.19: Example 277.

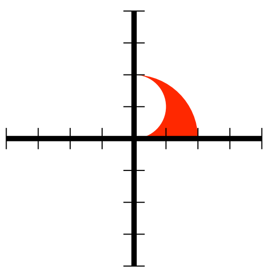

Figure 3.20: Example 278.

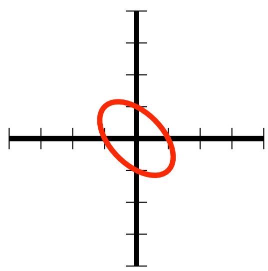

Figure 3.21: Example 279.

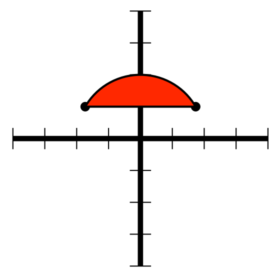

Figure 3.22: Example 280.

278 Example Evaluate $\int_R x\mathrm{d}A$, where R is the region bounded by the circles $x^2+y^2=4$ and $x^2+y^2=2y$.

Solution: ▸ *Observe that this is problem 3.5.23. Since $x^2+y^2=r^2$, the radius sweeps from $r^2=2r\sin\theta$ to $r^2=4$, that is, from $2\sin\theta$ to 2. The angle clearly sweeps from 0 to $\frac{\pi}{2}$. Thus the integral becomes*

$$\begin{aligned}\int_R x\mathrm{d}A &= \int_0^{\pi/2}\int_2^{2\sin\theta} r^2\cos\theta\mathrm{d}r\mathrm{d}\theta\\ &= \frac{1}{3}\int_0^{\pi/2}(8\cos\theta - 8\cos\theta\sin^3\theta)\mathrm{d}\theta\\ &= 2.\end{aligned}$$

◀

279 Example Find $\int_D e^{-x^2-xy-y^2}\,\mathrm{d}A$, where

$$D = \{(x,y) \in \mathbb{R}^2 : x^2+xy+y^2 \leq 1\}.$$

Solution: ▶ *Completing squares*

$$x^2+xy+y^2 = \left(x+\frac{y}{2}\right)^2 + \left(\frac{\sqrt{3}y}{2}\right)^2.$$

Put $U = x+\frac{y}{2}$, $V = \frac{\sqrt{3}y}{2}$. *The integral becomes*

$$\int_{\{x^2+xy+y^2\leq 1\}} e^{-x^2-xy-y^2}\mathrm{d}x\mathrm{d}y = \frac{2}{\sqrt{3}}\int_{\{U^2+V^2\leq 1\}} e^{-(U^2+V^2)}\mathrm{d}U\mathrm{d}V.$$

Passing to polar coordinates, the above equals

$$\frac{2}{\sqrt{3}}\int_0^{2\pi}\int_0^1 \rho e^{-\rho^2}\mathrm{d}\rho\mathrm{d}\theta = \frac{2\pi}{\sqrt{3}}(1-e^{-1}).$$

◀

280 Example Evaluate $\int_{\mathcal{R}} \frac{1}{(x^2+y^2)^{3/2}}\mathrm{d}A$ over the region $\left\{(x,y)\in\mathbb{R}^2 : x^2+y^2 \leq 4, y \geq 1\right\}$ (figure 3.22).

Solution: ▶ *The radius sweeps from* $r = \frac{1}{\sin\theta}$ *to* $r = 2$. *The desired integral is*

$$\begin{aligned}\int_{\mathcal{R}} \frac{1}{(x^2+y^2)^{3/2}}\mathrm{d}A &= \int_{\pi/6}^{5\pi/6}\int_{\csc\theta}^{2}\frac{1}{r^2}\mathrm{d}r\mathrm{d}\theta \\ &= \int_{\pi/6}^{5\pi/6}\left(\sin\theta - \frac{1}{2}\right)\mathrm{d}\theta \\ &= \sqrt{3}-\frac{\pi}{3}.\end{aligned}$$

◀

281 Example Evaluate $\int_R (x^3+y^3)\mathrm{d}A$ where R is the region bounded by the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$ and the first quadrant, $a > 0$ and $b > 0$.

Solution: ▶ *Put* $x = ar\cos\theta$, $y = br\sin\theta$. *Then*

$$x = ar\cos\theta \implies \mathrm{d}x = a\cos\theta\mathrm{d}r - ar\sin\theta\mathrm{d}\theta,$$

$$y = br\sin\theta \implies \mathrm{d}y = b\sin\theta\mathrm{d}r + br\cos\theta\mathrm{d}\theta,$$

whence

$$\mathrm{d}x\wedge\mathrm{d}y = (abr\cos^2\theta + abr\sin^2\theta)\mathrm{d}r\wedge\mathrm{d}\theta = abr\mathrm{d}r\wedge\mathrm{d}\theta.$$

Observe that on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1 \implies \frac{a^2r^2\cos^2\theta}{a^2}+\frac{b^2r^2\sin^2\theta}{b^2} = 1 \implies r = 1.$$

Thus the required integral is

$$\begin{aligned}\int_{R}(x^3+y^3)\mathrm{d}A &= \int_0^{\pi/2}\int_0^1 abr^4(\cos^3\theta+\sin^3\theta)\mathrm{d}r\mathrm{d}\theta\\ &= ab\left(\int_0^1 r^4\mathrm{d}r\right)\left(\int_0^{\pi/2}(a^3\cos^3\theta+b^3\sin^3\theta)\mathrm{d}\theta\right)\\ &= ab\left(\frac{1}{5}\right)\left(\frac{2a^3+2b^3}{3}\right)\\ &= \frac{2ab(a^3+b^3)}{15}.\end{aligned}$$

◀

Homework

Problem 3.7.1 Evaluate $\int_{\mathcal{R}} xy\mathrm{d}A$ where $\mathcal{R}$ is the region

$$\mathcal{R}=\left\{(x,y)\in\mathbb{R}^2 : x^2+y^2\le 16, x\ge 1, y\ge 1\right\},$$

as in the figure 3.23. Set up the integral in both Cartesian and polar coordinates.

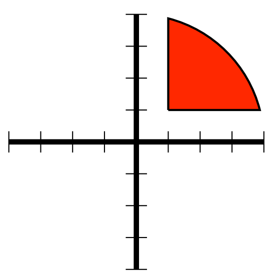

Figure 3.23: Problem 3.7.1.

Problem 3.7.2 Find

$$\int_D (x^2-y^2)\mathrm{d}A$$

where

$$D=\{(x,y)\in\mathbb{R}^2|(x-1)^2+y^2\le 1\}.$$

Problem 3.7.3 Find

$$\int_D \sqrt{xy}\mathrm{d}A$$

where

$$D=\{(x,y)\in\mathbb{R}^2|(x^2+y^2)^2\le 2xy\}.$$

Problem 3.7.4 Find $\int_D f\mathrm{d}A$ where

$$D=\{(x,y)\in\mathbb{R}^2 : b^2x^2+a^2y^2=a^2b^2, (a,b)\in]0;+\infty[\text{ fixed}\}$$

and $f(x,y)=x^3+y^3$.

Problem 3.7.5 Let $a > 0$ and $b > 0$. Prove that

$$\int_R \sqrt{\frac{a^2b^2 - a^2y^2 - b^2x^2}{a^2b^2 + a^2y^2 + b^2x^2}}\mathrm{d}A = \frac{\pi ab(\pi - 2)}{8},$$

where R is the region bounded by the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the first quadrant.

Problem 3.7.6 Prove that

$$\int_R \frac{y}{\sqrt{x^2 + y^2}}\mathrm{d}x\mathrm{d}y = \sqrt{2} - 1,$$

where

$$R = \{(x, y) \in \mathbb{R}^2 : 0 < x < 1, o < y < x^2\}.$$

Problem 3.7.7 Prove that the ellipse

$$(x - 2y + 3)^2 + (3x + 4y - 1)^2 = 4$$

bounds an area of $\frac{2\pi}{5}$.

Problem 3.7.8 Find

$$\int_D \sqrt{x^2 + y^2}\mathrm{d}A$$

where

$$D = \{(x, y) \in \mathbb{R}^2 | x \geq 0, y \geq 0, x^2 + y^2 \leq 1, x^2 + y^2 - 2y \geq 0\}.$$

Problem 3.7.9 Find $\int_D f\mathrm{d}A$ where

$$D = \{(x, y) \in \mathbb{R}^2 | y \geq 0, x^2 + y^2 - 2x \leq 0\}$$

and $f(x, y) = x^2y$.

Problem 3.7.10 Let $D = \{(x, y) \in \mathbb{R}^2 : x \geq 1, x^2 + y^2 \leq 4\}$. Find $\int_D x\mathrm{d}A$.

Problem 3.7.11 Find $\int_D f\mathrm{d}A$ where

$$D = \{(x, y) \in \mathbb{R}^2 | x \geq 1, x^2 + y^2 - 2x \leq 0\}$$

and $f(x, y) = \frac{1}{(x^2 + y^2)^2}$.

Problem 3.7.12 Let

$$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 - y \leq 0, x^2 + y^2 - x \leq 0\}.$$

Find the integral

$$\int_D (x + y)^2\mathrm{d}A.$$

Problem 3.7.13 Let $D = \{(x, y) \in \mathbb{R}^2 | y \leq x^2 + y^2 \leq 1\}$. Compute

$$\int_D \frac{\mathrm{d}A}{(1 + x^2 + y^2)^2}.$$

Problem 3.7.14 Evaluate

$$\int_{\{(x,y)\in\mathbb{R}^2 : x \geq 0, y \geq 0, x^4 + y^4 \leq 1\}} x^3y^3\sqrt{1 - x^4 - y^4}\,\mathrm{d}A$$

using $x^2 = \rho\cos\theta$, $y^2 = \rho\sin\theta$.

Problem 3.7.15 William Thompson (Lord Kelvin) is credited to have said: "A mathematician is someone to whom

$$\int_0^{+\infty} e^{-x^2} \mathrm{d}x = \frac{\sqrt{\pi}}{2}$$

is as obvious as twice two is four to you. Liouville was a mathematician." Prove that

$$\int_0^{+\infty} e^{-x^2} \mathrm{d}x = \frac{\sqrt{\pi}}{2}$$

by following these steps.

❶ Let $a > 0$ be a real number and put $D_a = \{(x, y) \in \mathbb{R}^2 | x^2 + y^2 \leq a^2\}$. Find

$$I_a = \int_{D_a} e^{-(x^2+y^2)} \,\mathrm{d}x\mathrm{d}y.$$

❷ Let $a > 0$ be a real number and put $\Delta_a = \{(x, y) \in \mathbb{R}^2 ||x| \leq a, |y| \leq a\}$. Let

$$J_a = \int_{\Delta_a} e^{-(x^2+y^2)} \,\mathrm{d}x\mathrm{d}y.$$

Prove that

$$I_a \leq J_a \leq I_{a\sqrt{2}}.$$

❸ Deduce that

$$\int_0^{+\infty} e^{-x^2} \mathrm{d}x = \frac{\sqrt{\pi}}{2}.$$

Problem 3.7.16 Let $D = \{(x, y) \in \mathbb{R}^2 : 4 \leq x^2 + y^2 \leq 16\}$ and $f(x, y) = \dfrac{1}{x^2 + xy + y^2}$. Find $\displaystyle\int_D f(x, y)\mathrm{d}A$.

Problem 3.7.17 Prove that every closed convex region in the plane of area $\geq \pi$ has two points which are two units apart.

Problem 3.7.18 In the xy-plane, if R is the set of points inside and on a convex polygon, let $D(x, y)$ be the distance from (x, y) to the nearest point R. Show that

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-D(x,y)} \,\mathrm{d}x\mathrm{d}y = 2\pi + L + A,$$

where L is the perimeter of R and A is the area of R.

3.8 Three-Manifolds

282 Definition A 3-*dimensional oriented manifold of* $\mathbb{R}^3$ is simply an open set (body) $V \in \mathbb{R}^3$, where the $+$ orientation is in the direction of the outward pointing normal to the body, and the $-$ orientation is in the direction of the inward pointing normal to the body. A general oriented 3-manifold is a union of open sets.

☞ *The region $-M$ has opposite orientation to M and*

$$\int_{-M} \omega = -\int_M \omega.$$

We will often write

$$\int_M f\mathrm{d}V$$

where $\mathrm{d}V$ *denotes the* volume element.

☞ *In this section, unless otherwise noticed, we will choose the positive orientation for the regions considered. This corresponds to using the volume form* $\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$.

Let $V \subseteq \mathbb{R}^3$. Given a function $f : V \to \mathbb{R}$, the integral

$$\int_V f\mathrm{d}V$$

is the sum of all the values of f restricted to V. In particular,

$$\int_V \mathrm{d}V$$

is the oriented volume of V.

283 Example Find

$$\int_{[0;1]^3} x^2 y e^{xyz}\, \mathrm{d}V.$$

Solution: ▶ *The integral is*

$$\begin{aligned}\int_0^1 \left(\int_0^1 \left(\int_0^1 x^2 y e^{xyz}\, \mathrm{d}z\right) \mathrm{d}y\right) \mathrm{d}x &= \int_0^1 \left(\int_0^1 x(e^{xy} - 1)\, \mathrm{d}y\right) \mathrm{d}x \\ &= \int_0^1 (e^x - x - 1)\mathrm{d}x \\ &= e - \frac{5}{2}.\end{aligned}$$

◀

284 Example Find $\int_R z\, \mathrm{d}V$ if

$$R = \{(x, y, z) \in \mathbb{R}^3 | x \geq 0, y \geq 0, z \geq 0, \sqrt{x} + \sqrt{y} + \sqrt{z} \leq 1\}.$$

Solution: ▶ *The integral is*

$$\begin{aligned}\int_R z\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \int_0^1 z\left(\int_0^{(1-\sqrt{z})^2} \left(\int_0^{(1-\sqrt{z}-\sqrt{x})^2} \mathrm{d}y\right) \mathrm{d}x\right) \mathrm{d}z \\ &= \int_0^1 z\left(\int_0^{(1-\sqrt{z})^2} (1 - \sqrt{z} - \sqrt{x})^2 \mathrm{d}x\right) \mathrm{d}z \\ &= \frac{1}{6}\int_0^1 z(1 - \sqrt{z})^4\, \mathrm{d}z \\ &= \frac{1}{840}.\end{aligned}$$

◀

285 Example Prove that

$$\int_V x\mathrm{d}V = \frac{a^2bc}{24},$$

where V is the tetrahedron

$$V = \left\{(x, y, z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, z \geq 0, \frac{x}{a} + \frac{y}{b} + \frac{z}{c} \leq 1\right\}.$$

Solution: ► *We have*

$$\begin{aligned}\int_V x\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \int_0^c\int_0^{b-bz/c}\int_0^{a-ay/b-az/c} x\mathrm{d}x\mathrm{d}y\mathrm{d}z \\ &= \frac{1}{2}\int_0^c\int_0^{b-bz/c}\left(a-\frac{ay}{b}-\frac{az}{c}\right)^2\mathrm{d}y\mathrm{d}z \\ &= \frac{1}{6}\int_0^c\frac{a^2\left(-z+c\right)^3 b}{c^3}\mathrm{d}x \\ &= \frac{a^2bc}{24}\end{aligned}$$

◄

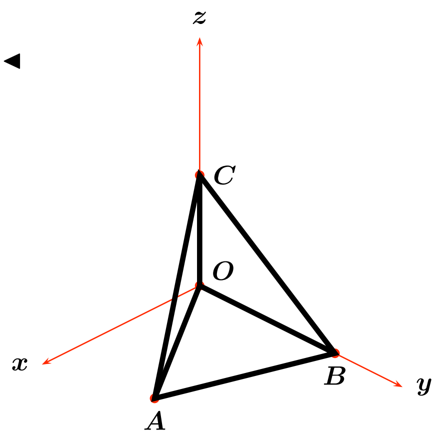

Figure 3.24: Problem 286.

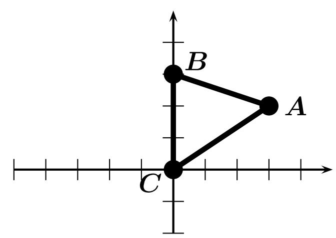

Figure 3.25: xy-projection.

286 Example Evaluate the integral $\int_S x\mathrm{d}V$ where S is the (unoriented) tetrahedron with vertices $(0,0,0)$, $(3,2,0)$, $(0,3,0)$, and $(0,0,2)$. See figure 3.24.

Solution: ► *A short computation shews that the plane passing through* $(3,2,0)$, $(0,3,0)$, *and* $(0,0,2)$ *has equation* $2x+6y+9z=18$. *Hence,* $0\le z\le\frac{18-2x-6y}{9}$. *We must now figure out the* xy *limits of integration. In figure 3.25 we draw the projection of the tetrahedron on the* xy *plane. The line passing through* AB *has equation* $y=-\frac{x}{3}+3$. *The line passing through* AC *has equation* $y=\frac{2}{3}x$.

We find, finally,

$$\begin{aligned}\int_S x\mathrm{d}V &= \int_0^3\int_{2x/3}^{3-x/3}\int_0^{(18-2x-6y)/9} x\mathrm{d}z\mathrm{d}y\mathrm{d}x \\ &= \int_0^3\int_{2x/3}^{3-x/3}\frac{18x-2x^2-6yx}{9}\mathrm{d}y\mathrm{d}x \\ &= \int_0^3\frac{18xy-2x^2y-3y^2x}{9}\bigg|_{2x/3}^{3-x/3}\mathrm{d}x \\ &= \int_0^3\left(\frac{x^3}{3}-2x^2+3x\right)\mathrm{d}x \\ &= \frac{9}{4}\end{aligned}$$

To solve this problem using Maple you may use the code below.

```
> with(Student[VectorCalculus]):
> int(x,[x,y,z]=Tetrahedron(<0,0,0>,<3,2,0>,<0,3,0>,<0,0,2>));
```

◄

287 Example Evaluate $\int_R xyz\mathrm{d}V$, where R is the solid formed by the intersection of the parabolic cylinder $z = 4 - x^2$, the planes $z = 0$, $y = x$, and $y = 0$. Use the following orders of integration:

1. $\mathrm{d}z\mathrm{d}x\mathrm{d}y$
2. $\mathrm{d}x\mathrm{d}y\mathrm{d}z$

Solution: ► *We must find the projections of the solid on the the coordinate planes.*

1. *With the order* $\mathrm{d}z\mathrm{d}x\mathrm{d}y$*, the limits of integration of* z *can only depend, if at all, on* x *and* y*. Given an arbitrary point in the solid, its lowest* z *coordinate is* 0 *and its highest one is on the cylinder, so the limits for* z *are from* $z = 0$ *to* $z = 4 - x^2$*. The projection of the solid on the* xy*-plane is the area bounded by the lines* $y = x$*,* $x = 2$*, and the* x *and* y *axes.*

$$
\begin{aligned}
\int_0^2\int_0^y\int_0^{4-x^2} xyz\mathrm{d}z\mathrm{d}x\mathrm{d}y &= \frac{1}{2}\int_0^2\int_0^y xy(4-x^2)^2\mathrm{d}x\mathrm{d}y \\
&= \frac{1}{2}\int_0^2\int_0^y y(16x - 8x^3 + x^5)\mathrm{d}x\mathrm{d}y \\
&= \int_0^2\left(4y^3 - y^5 + \frac{y^7}{12}\right)\mathrm{d}y \\
&= 8.
\end{aligned}
$$

2. *With the order* $\mathrm{d}x\mathrm{d}y\mathrm{d}z$*, the limits of integration of* x *can only depend, if at all, on* y *and* z*. Given an arbitrary point in the solid,* x *sweeps from the plane to* $x = 2$*, so the limits for* x *are from* $x = y$ *to* $x = \sqrt{4-z}$*. The projection of the solid on the* yz*-plane is the area bounded by* $z = 4 - y^2$*, and the* z *and* y *axes.*

$$
\begin{aligned}
\int_0^4\int_0^{\sqrt{4-z}}\int_y^2 xyz\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \frac{1}{2}\int_0^4\int_0^{\sqrt{4-z}}(4y - y^3)z\mathrm{d}y\mathrm{d}z \\
&= \int_0^4\left(2z - \frac{z^3}{8}\right)\mathrm{d}z \\
&= 8.
\end{aligned}
$$

◄

Homework

Problem 3.8.1 Compute $\int_E z\mathrm{d}V$ where E is the region in the first octant bounded by the planes $y + z = 1$ and $x + z = 1$.

Problem 3.8.2 Consider the solid S in the first octant, bounded by the parabolic cylinder $z = 2 - \frac{x^2}{2}$ and the planes $z = 0$, $y = x$, and $y = 0$. Prove that $\int_S xyz = \frac{2}{3}$ first by integrating in the order $\mathrm{d}z\mathrm{d}y\mathrm{d}x$, and then by integrating in the order $\mathrm{d}y\mathrm{d}x\mathrm{d}z$.

Problem 3.8.3 Evaluate the integrals $\int_R 1\mathrm{d}V$ and $\int_R x\mathrm{d}V$, where R is the tetrahedron with vertices at $(0,0,0)$, $(1,1,1)$, $(1,0,0)$, and $(0,0,1)$.

Problem 3.8.4 Compute $\int_E x\mathrm{d}V$ where E is the region in the first octant bounded by the plane $y = 3z$ and the cylinder $x^2 + y^2 = 9$.

Problem 3.8.5 Find $\int_D \frac{\mathrm{d}V}{(1+x^2z^2)(1+y^2z^2)}$ where

$$D = \{(x, y, z) \in \mathbb{R}^3 : 0 \le x \le 1, 0 \le y \le 1, z \ge 0\}.$$

3.9 Change of Variables

288 Example Find

$$\int_R (x+y+z)(x+y-z)(x-y-z)\mathrm{d}V,$$

where R is the tetrahedron bounded by the planes $x+y+z=0$, $x+y-z=0$, $x-y-z=0$, and $2x-z=1$.

Solution: ► *We make the change of variables*

$$u = x+y+z \implies \mathrm{d}u = \mathrm{d}x + \mathrm{d}y + \mathrm{d}z,$$

$$v = x+y-z \implies \mathrm{d}v = \mathrm{d}x + \mathrm{d}y - \mathrm{d}z,$$

$$w = x-y-z \implies \mathrm{d}w = \mathrm{d}x - \mathrm{d}y - \mathrm{d}z.$$

This gives

$$\mathrm{d}u \wedge \mathrm{d}v \wedge \mathrm{d}w = -4\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z.$$

These forms have opposite orientations, so we choose, say,

$$\mathrm{d}u \wedge \mathrm{d}w \wedge \mathrm{d}v = 4\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$$

which have the same orientation. Also,

$$2x - z = 1 \implies u + v + 2w = 2.$$

The tetrahedron in the xyz-coordinate frame is mapped into a tetrahedron bounded by $u = 0$, $v = 0$, $u+v+2w = 1$ in the uvw-coordinate frame. The integral becomes

$$\frac{1}{4}\int_0^2 \int_0^{1-v/2} \int_0^{2-v-2w} uvw \ \mathrm{d}u\mathrm{d}w\mathrm{d}v = \frac{1}{180}.$$

Consider a transformation to cylindrical coordinates

$$(x, y, z) = (\rho\cos\theta, \rho\sin\theta, z).$$

From what we know about polar coordinates

$$\mathrm{d}x \wedge \mathrm{d}y = \rho\mathrm{d}\rho \wedge \mathrm{d}\theta.$$

Since the wedge product of forms is associative,

$$\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = \rho\mathrm{d}\rho \wedge \mathrm{d}\theta \wedge \mathrm{d}z.$$

◄

289 Example Find $\int_R z^2 \mathrm{d}x\mathrm{d}y\mathrm{d}z$ if

$$R = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 \le 1, 0 \le z \le 1\}.$$

Solution: ► *The region of integration is mapped into*

$$\Delta = [0; 2\pi] \times [0; 1] \times [0; 1]$$

through a cylindrical coordinate change. The integral is therefore

$$\begin{aligned} \int_R f(x,y,z)\mathrm{d}x\mathrm{d}y\mathrm{d}z &= \left(\int_0^{2\pi} \mathrm{d}\theta\right)\left(\int_0^1 \rho\, \mathrm{d}\rho\right)\left(\int_0^1 z^2\, \mathrm{d}z\right) \\ &= \frac{\pi}{3}. \end{aligned}$$

◄

290 Example Evaluate $\int_D (x^2+y^2)\mathrm{d}x\mathrm{d}y\mathrm{d}z$ over the first octant region bounded by the cylinders $x^2+y^2=1$ and $x^2+y^2=4$ and the planes $z=0, z=1, x=0, x=y$.

Solution: ► *The integral is*

$$\int_0^1 \int_{\pi/4}^{\pi/2} \int_1^2 \rho^3 \mathrm{d}\rho\mathrm{d}\theta\mathrm{d}z = \frac{15\pi}{16}.$$

◄

291 Example Three long cylinders of radius R intersect at right angles. Find the volume of their intersection.

Solution: ► *Let V be the desired volume. By symmetry, $V = 2^4V'$, where*

$$V' = \int_{D'} \mathrm{d}x\mathrm{d}y\mathrm{d}z,$$

$$D' = \{(x,y,z) \in \mathbb{R}^3 : 0 \le y \le x, 0 \le z, x^2+y^2 \le R^2, y^2+z^2 \le R^2, z^2+x^2 \le R^2\}.$$

In this case it is easier to integrate with respect to z first. Using cylindrical coordinates

$$\Delta' = \left\{(\theta,\rho,z) \in \left[0; \frac{\pi}{4}\right] \times [0; R] \times [0; +\infty[, 0 \le z \le \sqrt{R^2-\rho^2\cos^2\theta}\right\}.$$

Now,

$$\begin{aligned} V' &= \int_0^{\pi/4}\left(\int_0^R\left(\int_0^{\sqrt{R^2-\rho^2\cos^2\theta}} \mathrm{d}z\right)\rho\mathrm{d}\rho\right)\mathrm{d}\theta \\ &= \int_0^{\pi/4}\left(\int_0^R \rho\sqrt{R^2-\rho^2\cos^2\theta}\mathrm{d}\rho\right)\mathrm{d}\theta \\ &= \int_0^{\pi/4} -\frac{1}{3\cos^2\theta}\left[(R^2-\rho^2\cos^2\theta)^{3/2}\right]_0^R \mathrm{d}\theta \\ &= \frac{R^3}{3}\int_0^{\pi/4} \frac{1-\sin^3\theta}{\cos^2\theta}\mathrm{d}\theta \\ \overset{=}{u=\cos\theta}\ & \frac{R^3}{3}\left([\tan\theta]_0^{\pi/4} + \int_1^{\frac{\sqrt{2}}{2}} \frac{1-u^2}{u^2}\mathrm{d}u\right) \\ &= \frac{R^3}{3}\left(1 - \left[u^{-1}+u\right]_1^{\frac{\sqrt{2}}{2}}\right) \\ &= \frac{\sqrt{2}-1}{\sqrt{2}}R^3. \end{aligned}$$

Finally

$$V = 16V' = 8(2-\sqrt{2})R^3.$$

◄

Consider now a change to spherical coordinates

$$x = \rho\cos\theta\sin\phi, \quad y = \rho\sin\theta\sin\phi, \quad z = \rho\cos\phi.$$

We have

$$\begin{array}{lll} \mathrm{d}x & = & \cos\theta\sin\phi\mathrm{d}\rho - \rho\sin\theta\sin\phi\mathrm{d}\theta + \rho\cos\theta\cos\phi\mathrm{d}\phi, \\ \mathrm{d}y & = & \sin\theta\sin\phi\mathrm{d}\rho + \rho\cos\theta\sin\phi\mathrm{d}\theta + \rho\sin\theta\cos\phi\mathrm{d}\phi, \\ \mathrm{d}z & = & \cos\phi\mathrm{d}\rho - \rho\sin\phi\mathrm{d}\phi. \end{array}$$

This gives

$$\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = -\rho^2\sin\phi\mathrm{d}\rho \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi.$$

From this derivation, the form $\mathrm{d}\rho \wedge \mathrm{d}\theta \wedge \mathrm{d}\phi$ is negatively oriented, and so we choose

$$\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = \rho^2\sin\phi\mathrm{d}\rho \wedge \mathrm{d}\phi \wedge \mathrm{d}\theta$$

instead.

292 Example Let $(a, b, c) \in]0; +\infty[^3$ be fixed. Find $\int_R xyz\, \mathrm{d}V$ if

$$R = \left\{(x, y, z) \in \mathbb{R}^3 : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1, x \geq 0, y \geq 0, z \geq 0\right\}.$$

Solution: ► *We use spherical coordinates, where*

$$(x, y, z) = (a\rho\cos\theta\sin\phi, b\rho\sin\theta\sin\phi, c\rho\cos\phi).$$

We have

$$\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z = abc\rho^2\sin\phi\mathrm{d}\rho \wedge \mathrm{d}\phi \wedge \mathrm{d}\rho.$$

The integration region is mapped into

$$\Delta = [0; 1] \times [0; \frac{\pi}{2}] \times [0; \frac{\pi}{2}].$$

The integral becomes

$$(abc)^2\left(\int_0^{\pi/2}\cos\theta\sin\theta\, \mathrm{d}\theta\right)\left(\int_0^1 \rho^5\, \mathrm{d}\rho\right)\left(\int_0^{\pi/2}\cos^3\phi\sin\phi\, \mathrm{d}\phi\right) = \frac{(abc)^2}{48}.$$

◄

293 Example Let $V = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq 9, 1 \leq z \leq 2\}$. Then

$$\begin{array}{lll} \displaystyle\int_V \mathrm{d}x\mathrm{d}y\mathrm{d}z & = & \displaystyle\int_0^{2\pi}\int_{\pi/2-\arcsin 2/3}^{\pi/2-\arcsin 1/3}\int_{1/\cos\phi}^{2/\cos\phi}\rho^2\sin\phi\, \mathrm{d}\rho\mathrm{d}\phi\mathrm{d}\theta \\ & = & \dfrac{63\pi}{4}. \end{array}$$

Homework

Problem 3.9.1 Consider the region $\mathcal{R}$ below the cone $z = \sqrt{x^2 + y^2}$ and above the paraboloid $z = x^2 + y^2$ for $0 \leq z \leq 1$. Set up integrals for the volume of this region in Cartesian, cylindrical and spherical coordinates. Also, find this volume.

Problem 3.9.2 Consider the integral $\int_{\mathcal{R}} x\mathrm{d}V$, where $\mathcal{R}$ is the region above the paraboloid $z = x^2 + y^2$ and under the sphere $x^2 + y^2 + z^2 = 4$. Set up integrals for the volume of this region in Cartesian, cylindrical and spherical coordinates. Also, find this volume.

Problem 3.9.3 Consider the region $\mathcal{R}$ bounded by the sphere $x^2 + y^2 + z^2 = 4$ and the plane $z = 1$. Set up integrals for the volume of this region in Cartesian, cylindrical and spherical coordinates. Also, find this volume.

Problem 3.9.4 Prove that the volume enclosed by the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is $\frac{4\pi abc}{3}$. Here $a > 0$, $b > 0$, $c > 0$.

Problem 3.9.5 Compute $\int_E y\mathrm{d}V$ where E is the region between the cylinders $x^2+y^2 = 1$ and $x^2 + y^2 = 4$, below the plane $x - z = -2$ and above the xy-plane.

Problem 3.9.6 Prove that

$$\int\limits_{\substack{x\geq 0, y\geq 0\\ x^2+y^2+z^2\leq R^2}} e^{-\sqrt{x^2+y^2+z^2}}\mathrm{d}V = \pi\left(2 - 2e^{-R} - 2Re^{-R} - R^2e^{-R}\right).$$

Problem 3.9.7 Compute $\int_E y^2z^2\mathrm{d}V$ where E is bounded by the paraboloid $x = 1 - y^2 - z^2$ and the plane $x = 0$.

Problem 3.9.8 Compute $\int_E z\sqrt{x^2 + y^2 + z^2}\mathrm{d}V$ where E is is the upper solid hemisphere bounded by the xy-plane and the sphere of radius 1 about the origin.

Problem 3.9.9 Compute the 4-dimensional integral

$$\iiiint\limits_{x^2+y^2+u^2+v^2\leq 1} e^{x^2+y^2+u^2+v^2}\mathrm{d}x\mathrm{d}y\mathrm{d}u\mathrm{d}v.$$

Problem 3.9.10 (Putnam Exam 1984) Find

$$\int\limits_R x^1y^9z^8(1 - x - y - z)^4\ \mathrm{d}x\mathrm{d}y\mathrm{d}z,$$

where

$$R = \{(x, y, z) \in \mathbb{R}^3 : x \geq 0, y \geq 0, z \geq 0, x + y + z \leq 1\}.$$

3.10 Surface Integrals

294 Definition A 2-*dimensional oriented manifold of* $\mathbb{R}^3$ is simply a smooth surface $D \in \mathbb{R}^3$, where the $+$ orientation is in the direction of the outward normal pointing away from the origin and the $-$ orientation is in the direction of the inward normal pointing towards the origin. A general oriented 2-manifold in $\mathbb{R}^3$ is a union of surfaces.

☞ *The surface $-\Sigma$ has opposite orientation to Σ and*

$$\int_{-\Sigma}\omega = -\int_{\Sigma}\omega.$$

☞ *In this section, unless otherwise noticed, we will choose the positive orientation for the regions considered. This corresponds to using the ordered basis*

$$\{\mathrm{d}y \wedge \mathrm{d}z,\ \mathrm{d}z \wedge \mathrm{d}x,\ \mathrm{d}x \wedge \mathrm{d}y\}.$$

295 Definition Let $f : \mathbb{R}^3 \to \mathbb{R}$. The integral of f over the smooth surface Σ (oriented in the positive sense) is given by the expression

$$\int_{\Sigma} f \left|\left|d^2\mathbf{x}\right|\right|.$$

Here

$$\left|\left|d^2\mathbf{x}\right|\right| = \sqrt{(dx \wedge dy)^2 + (dz \wedge dx)^2 + (dy \wedge dz)^2}$$

is the *surface area element.*

296 Example Evaluate $\int_{\Sigma} z \left|\left|d^2\mathbf{x}\right|\right|$ where Σ is the outer surface of the section of the paraboloid $z = x^2 + y^2, 0 \le z \le 1$.

Solution: ► *We parametrise the paraboloid as follows. Let* $x = u, y = v, z = u^2 + v^2$. *Observe that the domain* D *of* Σ *is the unit disk* $u^2 + v^2 \le 1$. *We see that*

$$dx \wedge dy = du \wedge dv,$$

$$dy \wedge dz = -2u du \wedge dv,$$

$$dz \wedge dx = -2v du \wedge dv,$$

and so

$$\left|\left|d^2\mathbf{x}\right|\right| = \sqrt{1 + 4u^2 + 4v^2} du \wedge dv.$$

Now,

$$\int_{\Sigma} z \left|\left|d^2\mathbf{x}\right|\right| = \int_{D} (u^2 + v^2)\sqrt{1 + 4u^2 + 4v^2} du dv.$$

To evaluate this last integral we use polar coordinates, and so

$$\begin{aligned} \int_{D} (u^2 + v^2)\sqrt{1 + 4u^2 + 4v^2} du dv &= \int_0^{2\pi}\int_0^1 \rho^3\sqrt{1 + 4\rho^2} d\rho d\theta \\ &= \frac{\pi}{12}(5\sqrt{5} + \frac{1}{5}). \end{aligned}$$

◄

297 Example Find the area of that part of the cylinder $x^2+y^2 = 2y$ lying inside the sphere $x^2+y^2+z^2 = 4$.

Solution: ► *We have*

$$x^2 + y^2 = 2y \iff x^2 + (y-1)^2 = 1.$$

We parametrise the cylinder by putting $x = \cos u, y - 1 = \sin u$, *and* $z = v$. *Hence*

$$dx = -\sin u du, \ dy = \cos u du, \ dz = dv,$$

whence

$$dx \wedge dy = 0, dy \wedge dz = \cos u du \wedge dv, dz \wedge dx = \sin u du \wedge dv,$$

and so

$$\begin{aligned} \left|\left|d^2\mathbf{x}\right|\right| &= \sqrt{(dx \wedge dy)^2 + (dz \wedge dx)^2 + (dy \wedge dz)^2} \\ &= \sqrt{\cos^2 u + \sin^2 u}\, du \wedge dv \\ &= du \wedge dv. \end{aligned}$$

The cylinder and the sphere intersect when $x^2+y^2=2y$ and $x^2+y^2+z^2=4$, that is, when $z^2=4-2y$, i.e. $v^2=4-2(1+\sin u)=2-2\sin u$. Also $0\le u\le\pi$. The integral is thus

$$\begin{aligned}\int_{\Sigma}\left|\left|\mathrm{d}^2\mathbf{x}\right|\right| &= \int_0^{\pi}\int_{-\sqrt{2-2\sin u}}^{\sqrt{2-2\sin u}}\mathrm{d}v\mathrm{d}u = \int_0^{\pi}2\sqrt{2-2\sin u}\mathrm{d}u\\ &= 2\sqrt{2}\int_0^{\pi}\sqrt{1-\sin u}\,\mathrm{d}u\\ &= 2\sqrt{2}\left(4\sqrt{2}-4\right).\end{aligned}$$

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298 Example Evaluate

$$\int_{\Sigma} x\mathrm{d}y\mathrm{d}z+(z^2-zx)\mathrm{d}z\mathrm{d}x-xy\mathrm{d}x\mathrm{d}y,$$

where Σ is the top side of the triangle with vertices at $(2,0,0)$, $(0,2,0)$, $(0,0,4)$.

Solution: ► *Observe that the plane passing through the three given points has equation $2x+2y+z=4$. We project this plane onto the coordinate axes obtaining*

$$\int_{\Sigma} x\mathrm{d}y\mathrm{d}z=\int_0^4\int_0^{2-z/2}(2-y-z/2)\mathrm{d}y\mathrm{d}z=\frac{8}{3},$$

$$\int_{\Sigma}(z^2-zx)\mathrm{d}z\mathrm{d}x=\int_0^2\int_0^{4-2x}(z^2-zx)\mathrm{d}z\mathrm{d}x=8,$$

$$-\int_{\Sigma}xy\mathrm{d}x\mathrm{d}y=-\int_0^2\int_0^{2-y}xy\mathrm{d}x\mathrm{d}y=-\frac{2}{3},$$

and hence

$$\int_{\Sigma} x\mathrm{d}y\mathrm{d}z+(z^2-zx)\mathrm{d}z\mathrm{d}x-xy\mathrm{d}x\mathrm{d}y=10.$$

◀

Homework

Problem 3.10.1 Evaluate $\int_{\Sigma} y\left|\left|\mathrm{d}^2\mathbf{x}\right|\right|$ where Σ is the surface $z=x+y^2, 0\le x\le 1, 0\le y\le 2$.

Problem 3.10.2 Consider the cone $z=\sqrt{x^2+y^2}$. Find the surface area of the part of the cone which lies between the planes $z=1$ and $z=2$.

Problem 3.10.3 Evaluate $\int_{\Sigma} x^2\left|\left|\mathrm{d}^2\mathbf{x}\right|\right|$ where Σ is the surface of the unit sphere $x^2+y^2+z^2=1$.

Problem 3.10.4 Evaluate $\int_S z\left|\left|\mathrm{d}^2\mathbf{x}\right|\right|$ over the conical surface $z=\sqrt{x^2+y^2}$ between $z=0$ and $z=1$.

Problem 3.10.5 You put a perfectly spherical egg through an egg slicer, resulting in n slices of identical height, but you forgot to peel it first! Shew that the amount of egg shell in any of the slices is the same. Your argument must use surface integrals.

Problem 3.10.6 Evaluate
$$\int_{\Sigma} xy\mathrm{d}y\mathrm{d}z - x^2\mathrm{d}z\mathrm{d}x + (x+z)\mathrm{d}x\mathrm{d}y,$$
where Σ is the top of the triangular region of the plane $2x+2y+z=6$ bounded by the first octant.

3.11 Green's, Stokes', and Gauss' Theorems

We are now in position to state the general Stoke's Theorem.

299 Theorem (General Stoke's Theorem) Let M be a smooth oriented manifold, having boundary ∂M. If ω is a differential form, then
$$\int_{\partial M} \omega = \int_M \mathrm{d}\omega.$$

In $\mathbb{R}^2$, if ω is a 1-form, this takes the name of *Green's Theorem.*

300 Example Evaluate $\oint_C (x-y^3)\mathrm{d}x + x^3\mathrm{d}y$ where C is the circle $x^2+y^2=1$.

Solution: ▶ *We will first use Green's Theorem and then evaluate the integral directly. We have*
$$\begin{aligned}\mathrm{d}\omega &= \mathrm{d}(x-y^3)\wedge\mathrm{d}x + \mathrm{d}(x^3)\wedge\mathrm{d}y \\ &= (\mathrm{d}x - 3y^2\mathrm{d}y)\wedge\mathrm{d}x + (3x^2\mathrm{d}x)\wedge\mathrm{d}y \\ &= (3y^2+3x^2)\mathrm{d}x\wedge\mathrm{d}y.\end{aligned}$$

The region M is the area enclosed by the circle $x^2+y^2=1$. Thus by Green's Theorem, and using polar coordinates,
$$\begin{aligned}\oint_C (x-y^3)\mathrm{d}x + x^3\mathrm{d}y &= \int_M (3y^2+3x^2)\mathrm{d}x\mathrm{d}y \\ &= \int_0^{2\pi}\int_0^1 3\rho^2\rho\mathrm{d}\rho\mathrm{d}\theta \\ &= \frac{3\pi}{2}.\end{aligned}$$

Aliter: *We can evaluate this integral directly, again resorting to polar coordinates.*
$$\begin{aligned}\oint_C (x-y^3)\mathrm{d}x + x^3\mathrm{d}y &= \int_0^{2\pi} (\cos\theta - \sin^3\theta)(-\sin\theta)\mathrm{d}\theta + (\cos^3\theta)(\cos\theta)\mathrm{d}\theta \\ &= \int_0^{2\pi} (\sin^4\theta + \cos^4\theta - \sin\theta\cos\theta)\mathrm{d}\theta.\end{aligned}$$

To evaluate the last integral, observe that $1 = (\sin^2\theta+\cos^2\theta)^2 = \sin^4\theta + 2\sin^2\theta\cos^2\theta + \cos^4\theta$, whence the integral equals
$$\begin{aligned}\int_0^{2\pi} (\sin^4\theta + \cos^4\theta - \sin\theta\cos\theta)\mathrm{d}\theta &= \int_0^{2\pi} (1 - 2\sin^2\theta\cos^2\theta - \sin\theta\cos\theta)\mathrm{d}\theta \\ &= \frac{3\pi}{2}.\end{aligned}$$

◀

In general, let
$$\omega = f(x,y)\mathrm{d}x + g(x,y)\mathrm{d}y$$

be a 1-form in $\mathbb{R}^2$. Then

$$\begin{aligned}
d\omega &= df(x,y)\wedge dx + dg(x,y)\wedge dy \\
&= \left(\frac{\partial}{\partial x}f(x,y)dx + \frac{\partial}{\partial y}f(x,y)dy\right)\wedge dx + \left(\frac{\partial}{\partial x}g(x,y)dx + \frac{\partial}{\partial y}g(x,y)dy\right)\wedge dy \\
&= \left(\frac{\partial}{\partial x}g(x,y) - \frac{\partial}{\partial y}f(x,y)\right)dx\wedge dy
\end{aligned}$$

which gives the classical Green's Theorem

$$\int\limits_{\partial M} f(x,y)dx + g(x,y)dy = \int\limits_{M}\left(\frac{\partial}{\partial x}g(x,y) - \frac{\partial}{\partial y}f(x,y)\right)dxdy.$$

In $\mathbb{R}^3$, if ω is a 2-form, the above theorem takes the name of *Gauß'* or the *Divergence Theorem.*

301 Example Evaluate $\int_S (x-y)dydz + zdzdx - ydxdy$ where S is the surface of the sphere

$$x^2 + y^2 + z^2 = 9$$

and the positive direction is the outward normal.

Solution: ► *The region M is the interior of the sphere $x^2+y^2+z^2=9$. Now,*

$$\begin{aligned}
d\omega &= (dx - dy)\wedge dy\wedge dz + dz\wedge dz\wedge dx - dy\wedge dx\wedge dy \\
&= dx\wedge dy\wedge dz.
\end{aligned}$$

The integral becomes

$$\begin{aligned}
\int\limits_M dxdydz &= \frac{4\pi}{3}(27) \\
&= 36\pi.
\end{aligned}$$

Aliter: *We could evaluate this integral directly. We have*

$$\int_\Sigma (x-y)dydz = \int_\Sigma xdydz,$$

since $(x,y,z)\mapsto -y$ is an odd function of y and the domain of integration is symmetric with respect to y. Now,

$$\begin{aligned}
\int_\Sigma xdydz &= \int_{-3}^{3}\int_0^{2\pi}|\rho|\sqrt{9-\rho^2}d\rho d\theta \\
&= 36\pi.
\end{aligned}$$

Also

$$\int_\Sigma zdzdx = 0,$$

since $(x,y,z)\mapsto z$ is an odd function of z and the domain of integration is symmetric with respect to z. Similarly

$$\int_\Sigma -ydxdy = 0,$$

since $(x,y,z)\mapsto -y$ is an odd function of y and the domain of integration is symmetric with respect to y. ◄

In general, let

$$\omega = f(x,y,z)dy\wedge dz + g(x,y,z)dz\wedge dx + h(x,y,z)dx\wedge dy$$

be a 2-form in $\mathbb{R}^3$. Then

$$\begin{aligned}
\mathrm{d}\omega &= \mathrm{d}f(x,y,z)\mathrm{d}y\wedge\mathrm{d}z+\mathrm{d}g(x,y,z)\mathrm{d}z\wedge\mathrm{d}x+\mathrm{d}h(x,y,z)\mathrm{d}x\wedge\mathrm{d}y\\
&= \left(\frac{\partial}{\partial x}f(x,y,z)\mathrm{d}x+\frac{\partial}{\partial y}f(x,y,z)\mathrm{d}y+\frac{\partial}{\partial z}f(x,y,z)\mathrm{d}z\right)\wedge\mathrm{d}y\wedge\mathrm{d}z\\
&\quad+\left(\frac{\partial}{\partial x}g(x,y,z)\mathrm{d}x+\frac{\partial}{\partial y}g(x,y,z)\mathrm{d}y+\frac{\partial}{\partial z}g(x,y,z)\mathrm{d}z\right)\wedge\mathrm{d}z\wedge\mathrm{d}x\\
&\quad+\left(\frac{\partial}{\partial x}h(x,y,z)\mathrm{d}x+\frac{\partial}{\partial y}h(x,y,z)\mathrm{d}y+\frac{\partial}{\partial z}h(x,y,z)\mathrm{d}z\right)\wedge\mathrm{d}x\wedge\mathrm{d}y\\
&= \left(\frac{\partial}{\partial x}f(x,y,z)+\frac{\partial}{\partial y}g(x,y,z)+\frac{\partial}{\partial z}h(x,y,z)\right)\mathrm{d}x\wedge\mathrm{d}y\wedge\mathrm{d}z,
\end{aligned}$$

which gives the classical Gauss's Theorem

$$\int_{\partial M} f(x,y,z)\mathrm{d}y\mathrm{d}z+g(x,y,z)\mathrm{d}z\mathrm{d}x+h(x,y,z)\mathrm{d}x\mathrm{d}y=\int_{M}\left(\frac{\partial}{\partial x}f(x,y,z)+\frac{\partial}{\partial y}g(x,y,z)+\frac{\partial}{\partial z}h(x,y,z)\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z.$$

Using classical notation, if

$$\overrightarrow{\mathrm{a}}=\begin{bmatrix}f(x,y,z)\\g(x,y,z)\\h(x,y,z)\end{bmatrix},\mathrm{d}\overrightarrow{\mathrm{S}}=\begin{bmatrix}\mathrm{d}y\mathrm{d}z\\\mathrm{d}z\mathrm{d}x\\\mathrm{d}x\mathrm{d}y\end{bmatrix},$$

then

$$\int_M(\nabla\bullet\overrightarrow{\mathrm{a}})\mathrm{d}V=\int_{\partial M}\overrightarrow{\mathrm{a}}\bullet\mathrm{d}\overrightarrow{\mathrm{S}}.$$

The classical Stokes' Theorem occurs when ω is a 1-form in $\mathbb{R}^3$.

302 Example Evaluate $\oint_C y\mathrm{d}x+(2x-z)\mathrm{d}y+(z-x)\mathrm{d}z$ where C is the intersection of the sphere $x^2+y^2+z^2=4$ and the plane $z=1$.

Solution: ► *We have*

$$\begin{aligned}
\mathrm{d}\omega &= (\mathrm{d}y)\wedge\mathrm{d}x+(2\mathrm{d}x-\mathrm{d}z)\wedge\mathrm{d}y+(\mathrm{d}z-\mathrm{d}x)\wedge\mathrm{d}z\\
&= -\mathrm{d}x\wedge\mathrm{d}y+2\mathrm{d}x\wedge\mathrm{d}y+\mathrm{d}y\wedge\mathrm{d}z+\mathrm{d}z\wedge\mathrm{d}x\\
&= \mathrm{d}x\wedge\mathrm{d}y+\mathrm{d}y\wedge\mathrm{d}z+\mathrm{d}z\wedge\mathrm{d}x.
\end{aligned}$$

Since on C, $z=1$, the surface Σ on which we are integrating is the inside of the circle $x^2+y^2+1=4$, i.e., $x^2+y^2=3$. Also, $z=1$ implies $\mathrm{d}z=0$ and so

$$\int_\Sigma \mathrm{d}\omega=\int_\Sigma \mathrm{d}x\mathrm{d}y.$$

Since this is just the area of the circular region $x^2+y^2\leq 3$, the integral evaluates to

$$\int_\Sigma \mathrm{d}x\mathrm{d}y=3\pi.$$

◄

In general, let

$$\omega=f(x,y,z)\mathrm{d}x+g(x,y,z)\mathrm{d}y++h(x,y,z)\mathrm{d}z$$

be a 1-form in $\mathbb{R}^3$. Then

$$\begin{aligned}
\mathrm{d}\omega &= \mathrm{d}f(x,y,z)\wedge \mathrm{d}x + \mathrm{d}g(x,y,z)\wedge \mathrm{d}y + \mathrm{d}h(x,y,z)\wedge \mathrm{d}z \\
&= \left(\frac{\partial}{\partial x}f(x,y,z)\mathrm{d}x + \frac{\partial}{\partial y}f(x,y,z)\mathrm{d}y + \frac{\partial}{\partial z}f(x,y,z)\mathrm{d}z\right)\wedge \mathrm{d}x \\
&\quad + \left(\frac{\partial}{\partial x}g(x,y,z)\mathrm{d}x + \frac{\partial}{\partial y}g(x,y,z)\mathrm{d}y + \frac{\partial}{\partial z}g(x,y,z)\mathrm{d}z\right)\wedge \mathrm{d}y \\
&\quad\quad + \left(\frac{\partial}{\partial x}h(x,y,z)\mathrm{d}x + \frac{\partial}{\partial y}h(x,y,z)\mathrm{d}y + \frac{\partial}{\partial z}h(x,y,z)\mathrm{d}z\right)\wedge \mathrm{d}z \\
&= \left(\frac{\partial}{\partial y}h(x,y,z) - \frac{\partial}{\partial z}g(x,y,z)\right)\mathrm{d}y\wedge \mathrm{d}z \\
&\quad\quad + \left(\frac{\partial}{\partial z}f(x,y,z) - \frac{\partial}{\partial x}h(x,y,z)\right)\mathrm{d}z\wedge \mathrm{d}x \\
&\quad\quad \left(\frac{\partial}{\partial x}g(x,y,z) - \frac{\partial}{\partial y}f(x,y,z)\right)\mathrm{d}x\wedge \mathrm{d}y
\end{aligned}$$

which gives the classical Stokes' Theorem

$$\begin{aligned}
\int_{\partial M} f(x,y,z)\mathrm{d}x + g(x,y,z)\mathrm{d}y + h(x,y,z)\mathrm{d}z & \\
= \int_M \left(\frac{\partial}{\partial y}h(x,y,z) - \frac{\partial}{\partial z}g(x,y,z)\right)\mathrm{d}y\mathrm{d}z & \\
+ \left(\frac{\partial}{\partial z}g(x,y,z) - \frac{\partial}{\partial x}f(x,y,z)\right)\mathrm{d}x\mathrm{d}y & \\
+ \left(\frac{\partial}{\partial x}h(x,y,z) - \frac{\partial}{\partial y}f(x,y,z)\right)\mathrm{d}x\mathrm{d}y. &
\end{aligned}$$

Using classical notation, if

$$\overrightarrow{\mathrm{a}} = \begin{bmatrix} f(x,y,z) \\ g(x,y,z) \\ h(x,y,z)\end{bmatrix}, \quad \mathrm{d}\overrightarrow{\mathrm{r}} = \begin{bmatrix}\mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z\end{bmatrix}, \quad \mathrm{d}\overrightarrow{\mathrm{S}} = \begin{bmatrix}\mathrm{d}y\mathrm{d}z \\ \mathrm{d}z\mathrm{d}x \\ \mathrm{d}x\mathrm{d}y\end{bmatrix},$$

then

$$\int_M (\nabla\times\overrightarrow{\mathrm{a}})\bullet \mathrm{d}\overrightarrow{\mathrm{S}} = \int_{\partial M}\overrightarrow{\mathrm{a}}\bullet \mathrm{d}\overrightarrow{\mathrm{r}}.$$

Homework

Problem 3.11.1 Evaluate $\oint_C x^3y\mathrm{d}x + xy\mathrm{d}y$ where C is the square with vertices at $(0,0)$, $(2,0)$, $(2,2)$ and $(0,2)$.

Problem 3.11.2 Consider the triangle $\triangle$ with vertices $A:(0,0)$, $B:(1,1)$, $C:(-2,2)$.

❶ If L_{PQ} denotes the equation of the line joining P and Q find L_{AB}, L_{AC}, and L_{BC}.

❷ Evaluate

$$\oint_{\triangle} y^2\mathrm{d}x + x\mathrm{d}y.$$

❸ Find

$$\int_{\mathscr{D}} (1-2y)\mathrm{d}x\wedge \mathrm{d}y$$

where $\mathscr{D}$ is the interior of $\triangle$.

Problem 3.11.3 Problems 1 through 4 refer to the differential form

$$\omega = x\mathrm{d}y \wedge \mathrm{d}z + y\mathrm{d}z \wedge \mathrm{d}x + 2z\mathrm{d}x \wedge \mathrm{d}y,$$

and the solid M whose boundaries are the paraboloid $z = 1 - x^2 - y^2$, $0 \leq z \leq 1$ and the disc $x^2 + y^2 \leq 1$, $z = 0$. The surface ∂M of the solid is positively oriented upon considering outward normals.

1. Prove that $\mathrm{d}\omega = 4\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$.
2. Prove that in Cartesian coordinates, $\displaystyle\int_{\partial M} \omega = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{1-x^2-y^2} 4\mathrm{d}z\mathrm{d}y\mathrm{d}x$.
3. Prove that in cylindrical coordinates, $\displaystyle\int_{M} \mathrm{d}\omega = \int_{0}^{2\pi}\int_{0}^{1}\int_{0}^{1-r^2} 4r\mathrm{d}z\mathrm{d}r\mathrm{d}\theta$.
4. Prove that $\displaystyle\int_{\partial M} x\mathrm{d}y\mathrm{d}z + y\mathrm{d}z\mathrm{d}x + 2z\mathrm{d}x\mathrm{d}y = 2\pi$.

Problem 3.11.4 Problems 1 through 4 refer to the box

$$M = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ 0 \leq z \leq 2\},$$

the upper face of the box

$$U = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq 1,\ 0 \leq y \leq 1,\ z = 2\},$$

the boundary of the box without the upper top $S = \partial M \setminus U$, and the differential form

$$\omega = (\arctan y - x^2)\mathrm{d}y \wedge \mathrm{d}z + (\cos x \sin z - y^3)\mathrm{d}z \wedge \mathrm{d}x + (2zx + 6zy^2)\mathrm{d}x \wedge \mathrm{d}y.$$

1. Prove that $\mathrm{d}\omega = 3y^2\mathrm{d}x \wedge \mathrm{d}y \wedge \mathrm{d}z$.
2. Prove that $\displaystyle\int_{\partial M} (\arctan y - x^2)\mathrm{d}y\mathrm{d}z + (\cos x \sin z - y^3)\mathrm{d}z\mathrm{d}x + (2zx + 6zy^2)\mathrm{d}x\mathrm{d}y = \int_0^2\int_0^1\int_0^1 3y^2\mathrm{d}x\mathrm{d}y\mathrm{d}z = 2$. Here the boundary of the box is positively oriented considering outward normals.
3. Prove that the integral on the upper face of the box is $\displaystyle\int_U (\arctan y - x^2)\mathrm{d}y\mathrm{d}z + (\cos x \sin z - y^3)\mathrm{d}z\mathrm{d}x + (2zx + 6zy^2)\mathrm{d}x\mathrm{d}y = \int_0^1\int_0^1 4x + 12y^2\mathrm{d}x\mathrm{d}y = 6$.
4. Prove that the integral on the open box is $\displaystyle\int_{\partial M \setminus U} (\arctan y - x^2)\mathrm{d}y\mathrm{d}z + (\cos x \sin z - y^3)\mathrm{d}z\mathrm{d}x + (2zx + 6zy^2)\mathrm{d}x\mathrm{d}y = -4$.

Problem 3.11.5 Problems 1 through 3 refer to a triangular surface T in $\mathbb{R}^3$ and a differential form ω. The vertices of T are at $A(6, 0, 0)$, $B(0, 12, 0)$, and $C(0, 0, 3)$. The boundary of of the triangle ∂T is oriented positively by starting at A, continuing to B, following to C, and ending again at A. The surface T is oriented positively by considering the top of the triangle, as viewed from a point far above the triangle. The differential form is

$$\omega = (2xz + \arctan e^x)\,\mathrm{d}x + (xz + (y+1)^y)\,\mathrm{d}y + \left(xy + \frac{y^2}{2} + \log(1 + z^2)\right)\mathrm{d}z.$$

1. Prove that the equation of the plane that contains the triangle T is $2x + y + 4z = 12$.
2. Prove that $\mathrm{d}\omega = y\mathrm{d}y \wedge \mathrm{d}z + (2x - y)\,\mathrm{d}z \wedge \mathrm{d}x + z\mathrm{d}x \wedge \mathrm{d}y$.
3. Prove that $\displaystyle\int_{\partial T} (2xz + \arctan e^x)\,\mathrm{d}x + (xz + (y+1)^y)\,\mathrm{d}y + \left(xy + \frac{y^2}{2} + \log(1 + z^2)\right)\mathrm{d}z = \int_0^3\int_0^{12-4z} y\mathrm{d}y\mathrm{d}z + \int_0^6\int_0^{3-x/2} 2x\mathrm{d}z\mathrm{d}x = 108$.

Problem 3.11.6 Use Green's Theorem to prove that

$$\int_\Gamma (x^2 + 2y^3)\mathrm{d}y = 16\pi,$$

where Γ is the circle $(x-2)^2 + y^2 = 4$. Also, prove this directly by using a path integral.

Problem 3.11.7 Let Γ denote the curve of intersection of the plane $x+y=2$ and the sphere $x^2-2x+y^2-2y+z^2=0$, oriented clockwise when viewed from the origin. Use Stoke's Theorem to prove that

$$\int_{\Gamma} y\mathrm{d}x + z\mathrm{d}y + x\mathrm{d}z = -2\pi\sqrt{2}.$$

Prove this directly by parametrising the boundary of the surface and evaluating the path integral.

Problem 3.11.8 Use Green's Theorem to evaluate

$$\oint_C (x^3 - y^3)\mathrm{d}x + (x^3 + y^3)\mathrm{d}y,$$

where C is the positively oriented boundary of the region between the circles $x^2+y^2=2$ and $x^2+y^2=4$.

A Answers and Hints

1.1.1 No. The zero vector $\overrightarrow{0}$, has magnitude but no direction.

1.1.2 We have $2\overrightarrow{BC} = \overrightarrow{BE} + \overrightarrow{EC}$. By Chasles' Rule $\overrightarrow{AC} = \overrightarrow{AE} + \overrightarrow{EC}$, and $\overrightarrow{BD} = \overrightarrow{BE} + \overrightarrow{ED}$. We deduce that

$$\overrightarrow{AC} + \overrightarrow{BD} = \overrightarrow{AE} + \overrightarrow{EC} + \overrightarrow{BE} + \overrightarrow{ED} = \overrightarrow{AD} + \overrightarrow{BC}.$$

But since $ABCD$ is a parallelogram, $\overrightarrow{AD} = \overrightarrow{BC}$. Hence

$$\overrightarrow{AC} + \overrightarrow{BD} = \overrightarrow{AD} + \overrightarrow{BC} = 2\overrightarrow{BC}.$$

1.1.4 We have $\overrightarrow{IA} = -3\overrightarrow{IB} \iff \overrightarrow{IA} = -3(\overrightarrow{IA} + \overrightarrow{AB}) = -3\overrightarrow{IA} - 3\overrightarrow{AB}$. Thus we deduce

$$\begin{aligned}\overrightarrow{IA} + 3\overrightarrow{IA} = -3\overrightarrow{AB} &\iff 4\overrightarrow{IA} = -3\overrightarrow{AB}\\ &\iff 4\overrightarrow{AI} = 3\overrightarrow{AB}\\ &\iff \overrightarrow{AI} = \frac{3}{4}\overrightarrow{AB}.\end{aligned}$$

Similarly

$$\begin{aligned}\overrightarrow{JA} = -\frac{1}{3}\overrightarrow{JB} &\iff 3\overrightarrow{JA} = -\overrightarrow{JB}\\ &\iff 3\overrightarrow{JA} = -\overrightarrow{JA} - \overrightarrow{AB}\\ &\iff 4\overrightarrow{JA} = -\overrightarrow{AB}\\ &\iff \overrightarrow{AJ} = \frac{1}{4}\overrightarrow{AB}\end{aligned}$$

.

Thus we take $\mathbf{I}$ such that $\overrightarrow{AI} = \frac{3}{4}\overrightarrow{AB}$ and $\mathbf{J}$ such that $\overrightarrow{AJ} = \frac{1}{4}\overrightarrow{AB}$.

Now

$$\begin{aligned}\overrightarrow{MA} + 3\overrightarrow{MB} &= \overrightarrow{MI} + \overrightarrow{IA} + 3\overrightarrow{IB}\\ &= 4\overrightarrow{MI} + \overrightarrow{IA} + 3\overrightarrow{IB}\\ &= 4\overrightarrow{MI},\end{aligned}$$

and

$$\begin{aligned}3\overrightarrow{MA} + \overrightarrow{MB} &= 3\overrightarrow{MJ} + 3\overrightarrow{JA} + \overrightarrow{MJ} + \overrightarrow{JB}\\ &= 4\overrightarrow{MJ} + 3\overrightarrow{JA} + \overrightarrow{JB}\\ &= 4\overrightarrow{MJ}.\end{aligned}$$

1.1.5

$$x + 1 = t = 2 - y \implies y = -x + 1.$$

1.1.6 $\alpha = \frac{4}{7}, \beta = \frac{3}{7}, l = \frac{4}{9}, m = \frac{2}{9}, n = \frac{1}{3}$.

1.1.8 [A]. $\overrightarrow{0}$, [B]. $\overrightarrow{0}$, [C]. $\overrightarrow{0}$, [D]. $\overrightarrow{0}$, [E]. $2\overrightarrow{c}(= 2\overrightarrow{d})$

1.3.2 Plainly,

$$\begin{bmatrix} a \\ b \end{bmatrix} = \frac{b-a}{2}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \frac{a+b}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

1.4.1 $a = -3, b = -\frac{1}{2}$.

1.4.2 The desired transformations are in figures A.1 through A.3.

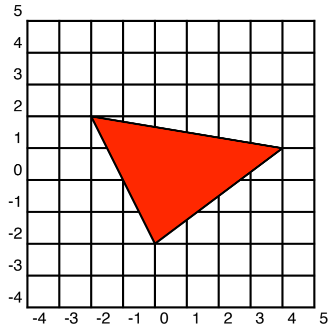

Figure A.1: Horizontal Stretch.

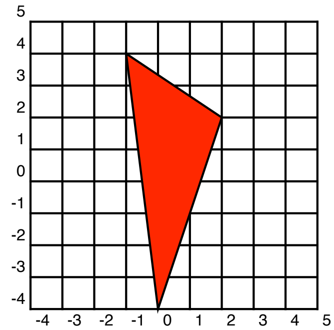

Figure A.2: Vertical Stretch.

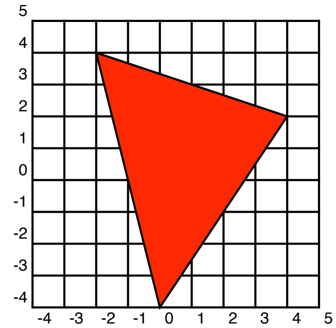

Figure A.3: Horizontal and Vertical Stretch.

1.4.3 The desired transformations are shewn in figures through A.4 A.7.

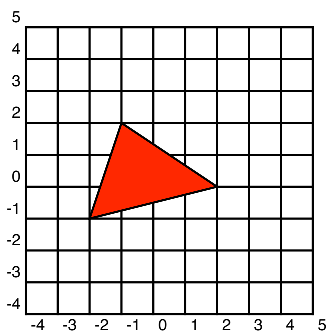

Figure A.4: Levogyrate rotation $\frac{\pi}{2}$ radians.

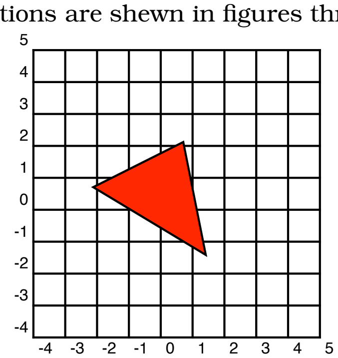

Figure A.5: Levogyrate rotation $\frac{\pi}{4}$ radians.

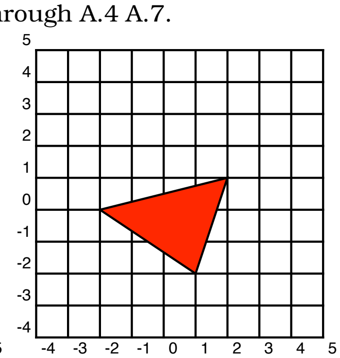

Figure A.6: Dextrogyrate rotation $\frac{\pi}{2}$ radians.

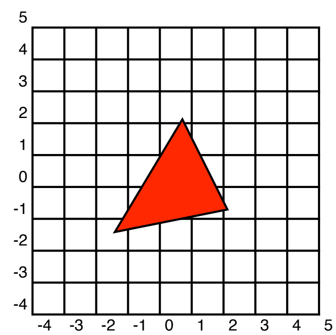

Figure A.7: Dextrogyrate rotation $\frac{\pi}{4}$ radians.

1.4.8 The transformations are shewn in figures A.8 through A.10.

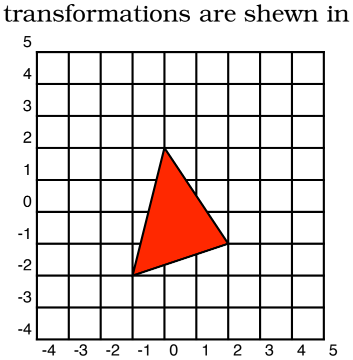

Figure A.8: Reflexion about the x-axis.

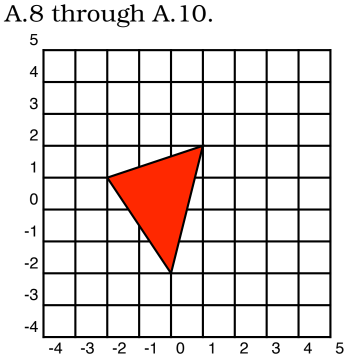

Figure A.9: Reflexion about the y-axis .

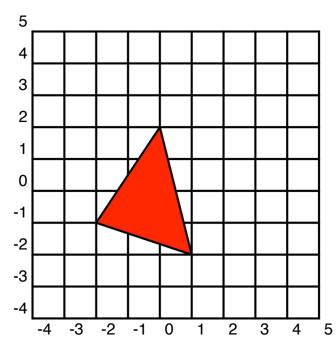

Figure A.10: Reflexion about the origin.

1.4.9 $\begin{bmatrix} a & b \\ c & -a \end{bmatrix}$, $bc = -a^2$

1.5.1 Upon solving the equations
$$-3a_1 + 2a_2 = 0, \qquad a_1^2 + a_2^2 = 13$$
we find $(a_1, a_2) = (2, 3)$ or $(a_1, a_2) = (-2, -3)$.

1.5.2 Since $\overrightarrow{a}\bullet\overrightarrow{b} = 0$, we have
$$\begin{aligned} ||\overrightarrow{a} + \overrightarrow{b}||^2 &= (\overrightarrow{a} + \overrightarrow{b})\bullet(\overrightarrow{a} + \overrightarrow{b}) \\ &= \overrightarrow{a}\bullet\overrightarrow{a} + 2\overrightarrow{a}\bullet\overrightarrow{b} + \overrightarrow{b}\bullet\overrightarrow{b} \\ &= \overrightarrow{a}\bullet\overrightarrow{a} + 0 + \overrightarrow{b}\bullet\overrightarrow{b} \\ &= ||\overrightarrow{a}||^2 + ||\overrightarrow{b}||^2, \end{aligned}$$
from where the desired result follows.

1.5.3 By the CBS Inequality,
$$(a^2 \cdot 1 + b^2 \cdot 1) \leq (a^4 + b^4)^{1/2}(1^2 + 1^2)^{1/2},$$
whence the assertion follows.

1.5.4 We have $\forall \overrightarrow{v} \in \mathbb{R}^2$, $\overrightarrow{v} \bullet (\overrightarrow{a} - \overrightarrow{b}) = 0$. In particular, choosing $\overrightarrow{v} = \overrightarrow{a} - \overrightarrow{b}$, we gather
$$(\overrightarrow{a} - \overrightarrow{b})\bullet(\overrightarrow{a} - \overrightarrow{b}) = ||\overrightarrow{a} - \overrightarrow{b}||^2 = 0.$$
But the norm of a vector is 0 if and only if the vector is the $\overrightarrow{0}$ vector. Therefore $\overrightarrow{a} - \overrightarrow{b} = \overrightarrow{0}$, i.e., $\overrightarrow{a} = \overrightarrow{b}$.

1.5.5 We have
$$\begin{aligned} ||\overrightarrow{u} + \overrightarrow{v}||^2 - ||\overrightarrow{u} - \overrightarrow{v}||^2 &= (\overrightarrow{u} + \overrightarrow{v})\bullet(\overrightarrow{u} + \overrightarrow{v}) - (\overrightarrow{u} - \overrightarrow{v})\bullet(\overrightarrow{u} - \overrightarrow{v}) \\ &= \overrightarrow{u}\bullet\overrightarrow{u} + 2\overrightarrow{u}\bullet\overrightarrow{v} + \overrightarrow{v}\bullet\overrightarrow{v} - (\overrightarrow{u}\bullet\overrightarrow{u} - 2\overrightarrow{u}\bullet\overrightarrow{v} + \overrightarrow{v}\bullet\overrightarrow{v}) \\ &= 4\overrightarrow{u}\bullet\overrightarrow{v}, \end{aligned}$$
giving the result.

1.5.6 A parametric equation for L_1 is
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ b_1 \end{pmatrix} + t\begin{bmatrix} 1 \\ m_1 \end{bmatrix}.$$
A parametric equation for L_2 is
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ b_2 \end{pmatrix} + t\begin{bmatrix} 1 \\ m_2 \end{bmatrix}.$$
The lines are perpendicular if and only if, according to Corollary 22,
$$\begin{bmatrix} 1 \\ m_1 \end{bmatrix} \bullet \begin{bmatrix} 1 \\ m_2 \end{bmatrix} = 0 \iff 1 + m_1m_2 = 0 \iff m_1m_2 = -1.$$

1.5.7 The line L has a parametric equation
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + t\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$
Let L' have parametric equation
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix} + t\begin{bmatrix} 1 \\ m \end{bmatrix}.$$
We need the angle between $\begin{bmatrix} 1 \\ m \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ to be $\frac{\pi}{6}$ and so by Theorem 21,
$$1 - m = \sqrt{1 + m^2}\sqrt{2}\cos\frac{\pi}{6} \implies m = -2 \pm \sqrt{3}.$$
This gives two possible values for the slope of L'. Now, since L' must pass through $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$
$$y = (-2 \pm \sqrt{3})x + b \implies 2 = (-2 \pm \sqrt{3})(-1) + b \implies b = \pm\sqrt{3}$$
and the lines are $y = (-2 + \sqrt{3})x + \sqrt{3}$ and $y = (-2 - \sqrt{3})x - \sqrt{3}$, respectively.

1.5.8 We must prove that $\vec{a}\bullet\vec{w} = 0$. Using the distributive law for the dot product,

$$\begin{aligned}\left(\vec{v} - \frac{\vec{v}\bullet\vec{w}}{\left\|\vec{w}\right\|^2}\vec{w}\right)\bullet\vec{w} &= \vec{v}\bullet\vec{w} - \frac{\vec{v}\bullet\vec{w}}{\left\|\vec{w}\right\|^2}\vec{w}\bullet\vec{w} \\ &= \vec{v}\bullet\vec{w} - \frac{\vec{v}\bullet\vec{w}}{\left\|\vec{w}\right\|^2}\left\|\vec{w}\right\|^2 \\ &= \vec{v}\bullet\vec{w} - \vec{v}\bullet\vec{w} \\ &= 0.\end{aligned}$$

1.6.1 We have $y - x = 4t \implies t = \frac{y-x}{4}$ and so

$$x = \left(\frac{y-x}{4}\right)^3 - 2\left(\frac{y-x}{4}\right)$$

is the Cartesian equation sought.

1.6.2 Observe that for $t \neq \{0, -1\}$,

$$\frac{y}{x} = t \implies x = \frac{\left(\frac{y}{x}\right)^2}{1 + \left(\frac{y}{x}\right)^5} \implies x = \frac{y^2x^3}{x^5 + y^5} \implies x^5 + y^5 = x^2y^2.$$

If $t = 0$, then $x = 0, y = 0$ and our Cartesian equation agrees. What happens as $t \to -1$?

1.6.3

1. $ay - cx = ad - bc$, this is a straight line with positive slope.
2. $-1 \leq x \leq 1, y = 0$, this is the line segment on the plane joining $(-1, 0)$ to $(1, 0)$.
3. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, $x > 0$. This is one branch of a hyperbola.
4. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. This is a hyperbola.

1.6.4 We may simply give the trivial parametrisation: $x = t$, $y = \log\cos t$, $0 \leq t \leq \frac{\pi}{3}$. Then

$$(\mathrm{d}x)^2 + (\mathrm{d}y)^2 = (1 + \tan^2 t)(\mathrm{d}t)^2 = \sec^2 t(\mathrm{d}t)^2.$$

Hence the arc length is

$$\int_0^{\pi/3} \sec t\mathrm{d}t = \log(2 + \sqrt{3}).$$

1.6.5 Observe that $y = 2x + 1$, so the trace is part of this line. Since in the interval $[0; 4\pi]$, $-1 \leq \sin t \leq 1$, we want the portion of the line $y = 2x + 1$ with $-1 \leq x \leq 1$ (and, thus $-1 \leq y \leq 3$). The curve starts at the middle point $(0, 1)$ (at $t = 0$), reaches the high point $(1, 3)$ at $t = \frac{\pi}{2}$, reaches its low point $(-1, 1)$ at $t = \frac{3\pi}{2}$, reaches its high point $(1, 3)$ again at $t = \frac{5\pi}{2}$, it goes to its low point $(-1, 1)$ at $t = \frac{7\pi}{2}$, and finishes in the middle point $(0, 1)$ when $t = 4\pi$.

1.6.6 First observe that $\sqrt{x} + \sqrt{y} = 1$ demands $x \in [0; 1]$ and $[0; 1]$. Again, one can give many parametrisations. One is $x = t^2$, $y = (1 - t)^2$, $t \in [0; 1]$. This gives

$$\sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = \sqrt{4t^2 + (2 - 2t)^2}\ \mathrm{d}t = 2\sqrt{2t^2 - 2t + 1}\ \mathrm{d}t = \frac{2}{\sqrt{2}}\sqrt{4\left(t - \frac{1}{2}\right)^2 + 1}\ \mathrm{d}t.$$

To integrate

$$\frac{2}{\sqrt{2}}\int_0^1 \sqrt{4\left(t - \frac{1}{2}\right)^2 + 1}\ \mathrm{d}t,$$

we now use the trigonometric substitution

$$2\left(t - \frac{1}{2}\right) = \tan\theta \implies \mathrm{d}t = \frac{1}{2}\sec^2\theta\mathrm{d}\theta.$$

The integral thus becomes

$$\sqrt{2}\int_0^{\pi/4}\sec^3\theta \mathrm{d}\theta,$$

the famous secant cube integral, which is a standard example of integration by parts where you "solve" for the integral. (You write $\int \sec\theta \mathrm{d}\tan\theta = \tan\theta\sec\theta - \int \tan\theta \mathrm{d}\sec\theta$, etc.) I will simply quote it, as I assume most of you have seen it, and it appears in most Calculus texts:

$$\begin{aligned}\sqrt{2}\int_0^{\pi/4}\sec^3\theta \mathrm{d}\theta &= \sqrt{2}\left(\frac{1}{2}\sec\theta\tan\theta - \frac{1}{2}\log|\sec\theta+\tan\theta|\right)\Big|_0^{\pi/4}\\ &= \sqrt{2}\left(\frac{1}{2}\cdot\sqrt{2}+\frac{1}{2}\log(\sqrt{2}+1)\right)\\ &= \frac{1}{\sqrt{2}}\log(\sqrt{2}+1)+1.\end{aligned}$$

1.6.7 First notice that $x = 1 \implies t^3+1 = 1 \implies t = 0$ and $x = 2 \implies t^3+1 = 2 \implies t = 1$. The area under the graph is

$$\int_{t=0}^{t=1} y\mathrm{d}x = \int_{t=0}^{t=1}(1-t^2)\mathrm{d}(t^3+1) = \int_{t=0}^{t=1}3t^2(1-t^2)\mathrm{d}t = \frac{2}{5}.$$

1.6.8 Observe that

$$\mathrm{d}x = 6t\mathrm{d}t; \quad \mathrm{d}y = 6t^2\mathrm{d}t \implies \sqrt{(\mathrm{d}x)^2+(\mathrm{d}y)^2} = 6t\sqrt{1+t^2}\ \mathrm{d}t,$$

and so the arc length is

$$\int_{t=0}^{t=1}\sqrt{(\mathrm{d}x)^2+(\mathrm{d}y)^2} = \int_{t=0}^{t=1}6t\sqrt{1+t^2}\ \mathrm{d}t = 4\sqrt{2}-2.$$

1.6.9 Observe that

$$\mathrm{d}x = t\mathrm{d}t, \qquad \mathrm{d}y = \sqrt{2t+1}\mathrm{d}t \implies \sqrt{(\mathrm{d}x)^2+(\mathrm{d}y)^2} = \sqrt{t^2+(\sqrt{2t+1})^2}\mathrm{d}t = \sqrt{t^2+2t+1}\mathrm{d}t = (t+1)\mathrm{d}t.$$

Hence, the arc length is

$$\int_{-1/2}^{1/2}(1+t)\mathrm{d}t = \left(t+\frac{t^2}{2}\right)\Big|_{-1/2}^{1/2} = 1.$$

1.6.10 Observe that the parametrisation traverses the curve once clockwise if $t \in [0; 2\pi]$. The area is given by

$$\begin{aligned}\frac{1}{2}\oint_\Gamma \det\begin{bmatrix}x & \mathrm{d}x\\ y & \mathrm{d}y\end{bmatrix} &= \frac{1}{2}\oint x\mathrm{d}y - y\mathrm{d}x\\ &= \frac{4}{2}\int_{\pi/2}^{0}(\sin^3 t(-\sin t(1+\sin^2 t)+2\sin t\cos^2 t)\\ &\qquad -\cos t(1+\sin^2 t)(3\sin^2 t\cos t))\mathrm{d}t\\ &= 2\int_{\pi/2}^{0}(-3\sin^2 t+\sin^4 t)\mathrm{d}t\\ &= 2\int_{\pi/2}^{0}\left(-\frac{9}{8}+\cos 2t+\frac{1}{8}\cos 4t\right)\mathrm{d}t\\ &= \frac{9\pi}{8}.\end{aligned}$$

1.6.11 Using the quotient rule,

$$\mathrm{d}x = \frac{3(1+t^3)-3t^2(3t)}{(1+t^3)^2}\cdot\mathrm{d}t = \frac{3-6t^3}{(1+t^3)^2}\cdot\mathrm{d}t \implies y\mathrm{d}x = \frac{9t^2-18t^5}{(1+t^3)^3}\cdot\mathrm{d}t$$

and

$$\mathrm{d}y = \frac{6t(1+t^3)-3t^2(3t^2)}{(1+t^3)^2}\cdot\mathrm{d}t = \frac{6t-3t^4}{(1+t^3)^2}\cdot\mathrm{d}t \implies x\mathrm{d}y = \frac{18t^2-9t^5}{(1+t^3)^3}\cdot\mathrm{d}t$$

Hence

$$x\mathrm{d}y - y\mathrm{d}x = \frac{18t^2-9t^5}{(1+t^3)^3}\cdot\mathrm{d}t - \frac{9t^2-18t^5}{(1+t^3)^3}\cdot\mathrm{d}t = \frac{9t^2+9t^5}{(1+t^3)^3}\cdot\mathrm{d}t = \frac{9t^2(1+t^3)}{(1+t^3)^3}\cdot\mathrm{d}t = \frac{9t^2}{(1+t^3)^2}\cdot\mathrm{d}t.$$

Observe that when $t = 0$ then $x = y = 0$. As $t \to +\infty$, then $x \to 0$ and $y \to 0$. Hence to obtain the loop Using integration by substitution ($u = 1+t^3$ and $\mathrm{d}u = 3t^2\mathrm{d}t$) , the area is given by

$$\frac{1}{2}\int_0^{+\infty}\frac{9t^2}{(1+t^3)^2}\cdot\mathrm{d}t = \frac{3}{2}\int_0^{+\infty}\frac{3t^2}{(1+t^3)^2}\mathrm{d}t = \frac{3}{2}\int_1^{+\infty}\frac{\mathrm{d}u}{u^2} = \frac{3}{2}.$$

☞ *A shorter way of obtaining $x\mathrm{d}y - y\mathrm{d}x$ would have been to argue that $x\mathrm{d}y - y\mathrm{d}x = x^2\mathrm{d}\left(\frac{y}{x}\right) = \frac{9t^2}{(1+t^3)^2}\mathrm{d}t$.*

1.6.12 See figure 1.58. Let θ be the angle (in radians) of rotation of the circle, and let C be the centre of the circle. At $\theta = 0$ the centre of the circle is at $(0, \rho)$, and $P = (0, \rho - d)$. Suppose the circle is displaced towards the right, making the point P to rotate an angle of θ radians. Then the centre of the circle has displaced $r\theta$ units horizontally, and so is now located at $(\rho\theta, \rho)$. The polar coordinates of the point P are $(d\sin\theta; d\cos\theta)$, in relation to the centre of the circle (notice that the circle moves clockwise). The point P has moved $x = \rho\theta - d\sin\theta$ horizontal units and $y = \rho - d\cos\theta$ units. This is the desired parametrisation.

1.6.14 We have

$$\mathrm{d}x = e^t(\cos t - \sin t)\mathrm{d}t,\ \mathrm{d}y = e^t(\sin t + \cos t)\mathrm{d}t \implies \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = e^t\sqrt{(\cos t - \sin t)^2 + (\sin t + \cos t)^2}\mathrm{d}t = \sqrt{2}e^t\mathrm{d}t.$$

The arc length is thus

$$\int_0^{\pi} \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = \sqrt{2}\int_0^{\pi} e^t\mathrm{d}t = \sqrt{2}(e^{\pi} - 1).$$

1.6.15 Choose coordinates so that the origin is at the position of the gun, the y-axis is vertical, and the airplane is on a point with coordinates (u, h) with $u \geq 0$.

If the gun were fired at $t = 0$, then

$$x = Vt\cos a; \quad y = Vt\sin a - \frac{gt^2}{2},$$

where a is the angle of elevation, t is the time and g is the acceleration due to gravity. Since we know that the shell strikes the plane, we must have

$$u = Vt\cos a; \quad h = Vt\sin a - \frac{gt^2}{2},$$

whence

$$u^2 + \left(h + \frac{1}{2}gt^2\right)^2 = V^2t^2,$$

and thus

$$\frac{g^2t^4}{4} + (gh - V^2)^2t^2 + h^2 + u^2 = 0.$$

The quadratic equation in t^2 has a real root if

$$(gh - V^2)^2 \geq g^2(h^2 + u^2) \implies g^2u^2 \leq V^2(V^2 - 2gh),$$

from where the assertion follows.

1.6.16 Suppose the parabolas have a point of contact $P = (4px_0^2, 4px_0)$. By symmetry, the vertex V of the rolling parabola is the reflexion of the origin about the line tangent to their point of contact. The slope of the tangent at P is $\frac{1}{2x_0}$, from where the equation of the tangent is

$$y = \frac{x}{2x_0} + 2px_0.$$

The line normal to this line and passing through the origin is hence

$$y = -2xx_0,$$

and so the lines intersect at

$$\left(-\frac{4px_0^2}{1+4x_0^2}, \frac{8px_0^3}{1+4x_0^2}\right),$$

from where

$$V = \left(-\frac{8px_0^2}{1+4x_0^2}, \frac{16px_0^3}{1+4x_0^2}\right) := (x(t), y(t)).$$

As $-2x_0x(t) = y(t)$, eliminating t yields

$$x = -\frac{2p\left(\frac{y}{x}\right)^2}{1 + \left(\frac{y}{x}\right)^2}$$

or

$$(x^2 + y^2)x + 2py^2 = 0,$$

giving the equation of the locus.

1.7.1 Observe that, in general,

$$\begin{aligned}\left\|\vec{a}-\vec{b}\right\|^2+\left\|\vec{a}+\vec{b}\right\|^2 &= \left\|\vec{a}\right\|^2-2\vec{a}\bullet\vec{b}+\left\|\vec{b}\right\|^2+\left\|\vec{a}\right\|^2+2\vec{a}\bullet\vec{b}+\left\|\vec{b}\right\|^2\\ &= 2\left\|\vec{a}\right\|^2+2\left\|\vec{b}\right\|^2,\end{aligned}$$

whence

$$\left\|\vec{a}-\vec{b}\right\| = \sqrt{2\left\|\vec{a}\right\|^2+2\left\|\vec{b}\right\|^2-\left\|\vec{a}+\vec{b}\right\|}=\sqrt{2(13)^2+2(19)^2-(24)^2}=22.$$

1.7.2 $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t\begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}.$

1.7.3 The vectorial form of the equation of the line is

$$\vec{r} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}.$$

Since the line follows the direction of $\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$, this means that $\begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix}$ is normal to the plane, and thus the equation of the desired plane is

$$(x-1)-2(y-1)-(z-1)=0.$$

1.7.4 Observe that $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ (as $0 = 2(0) = 3(0)$) is on the line, and hence on the plane. Thus the vector

$$\begin{bmatrix} 1-0 \\ -1-0 \\ -1-0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

lies on the plane. Now, if $x = 2y = 3z = t$, then $x = t, y = t/2, z = t/3$. Hence, the vectorial form of the equation of the line is

$$\vec{r} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 1/2 \\ 1/3 \end{bmatrix} = t\begin{bmatrix} 1 \\ 1/2 \\ 1/3 \end{bmatrix}.$$

This means that $\begin{bmatrix} 1 \\ 1/2 \\ 1/3 \end{bmatrix}$ also lies on the plane, and thus

$$\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 1/2 \\ 1/3 \end{bmatrix} = \begin{bmatrix} 1/6 \\ -4/3 \\ 3/2 \end{bmatrix}$$

is normal to the plane. The desired equation is thus

$$\frac{1}{6}x - \frac{4}{3}y + \frac{3}{2}z = 0.$$

1.7.5 The set B can be decomposed into the following subsets:

❶ The set A itself, of volume abc.

❷ Two $a \times b \times 1$ bricks, two $b \times c \times 1$ bricks, and two $c \times a \times 1$ bricks,

❸ Four quarter-cylinders of length a and radius 1, four quarter-cylinders of length b and radius 1, and four quarter-cylinders of length c and radius 1,

❹ Eight eighth-of-spheres of radius 1.

Thus the required formula for the volume is

$$abc + 2(ab + bc + ca) + \pi(a + b + c) + \frac{4\pi}{3}.$$

1.7.6 We have,

$$\left\|\vec{a} + \vec{b} + \vec{c}\right\|^2 = (\vec{a} + \vec{b} + \vec{c}) \bullet (\vec{a} + \vec{b} + \vec{c}) = \left\|\vec{a}\right\|^2 + \left\|\vec{b}\right\|^2 + \left\|\vec{c}\right\|^2 + 2(\vec{a}\bullet\vec{b} + \vec{b}\bullet\vec{c} + \vec{c}\bullet\vec{a}),$$

from where we deduce that

$$\vec{a}\bullet\vec{b} + \vec{b}\bullet\vec{c} + \vec{c}\bullet\vec{a} = \frac{-3^2 - 4^2 - 5^2}{2} = -25.$$

1.7.7 A vector normal to the plane is $\begin{bmatrix} a \\ a^2 \\ a^2 \end{bmatrix}$. The line sought has the same direction as this vector, thus the equation of the line is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} a \\ a^2 \\ a^2 \end{bmatrix}, \quad t \in \mathbb{R}.$$

1.7.8 Put $ax = by = cz = t$, so $x = t/a; y = t/b; z = t/c$. The parametric equation of the line is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t\begin{bmatrix} 1/a \\ 1/b \\ 1/c \end{bmatrix}, \quad t \in \mathbb{R}.$$

Thus the vector $\begin{bmatrix} 1/a \\ 1/b \\ 1/c \end{bmatrix}$ is perpendicular to the plane. Therefore, the equation of the plane is

$$\begin{bmatrix} 1/a \\ 1/b \\ 1/c \end{bmatrix} \bullet \begin{bmatrix} x-1 \\ y-1 \\ z-1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},$$

or

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}.$$

We may also write this as

$$bcx + cay + abz = ab + bc + ca.$$

1.7.9 The vector $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ is perpendicular to the plane. Hence, the shortest distance from $(1, 2, 3)$ is obtained by the perpendicular line to the plane that pierces the plane, this perpendicular line to the plane has equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} + t\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \implies x = 1 + t, y = 2 - t, z = 3 + t.$$

The intersection of the line and the plane occurs when

$$1 + t - (2 - t) + (3 + t) = 1 \implies t = -\frac{1}{3}.$$

The closest point on the plane to $(1, 2, 3)$ is therefore $\left(\frac{2}{3}, \frac{7}{3}, \frac{8}{3}\right)$, and the distance sought is

$$\sqrt{(1 - \frac{2}{3})^2 + (2 - \frac{7}{3})^2 + (3 - \frac{8}{3})^2} = \frac{\sqrt{3}}{3}.$$

1.7.10 If the lines intersected, there would be a value t' for which

$$\begin{pmatrix}1\\1\\1\end{pmatrix}+t'\begin{bmatrix}2\\1\\1\end{bmatrix}=\begin{pmatrix}0\\0\\1\end{pmatrix}+t'\begin{bmatrix}2\\-1\\1\end{bmatrix}\implies\begin{bmatrix}1-0\\1-0\\1-1\end{bmatrix}=t'\begin{bmatrix}2-2\\-1-1\\1-1\end{bmatrix}\implies\begin{bmatrix}1\\1\\0\end{bmatrix}=t'\begin{bmatrix}0\\-2\\0\end{bmatrix},$$

which is clearly impossible, and so the lines are skew. Let θ be the angle between them. Then

$$\cos\theta=\frac{2\cdot 2+1\cdot(-1)+1\cdot 1}{\sqrt{(2)^2+(1)^2+(1)^2}\sqrt{(2)^2+(-1)^2+(1)^2}}=\frac{4}{\sqrt{6}\sqrt{6}}\implies\theta=\arccos\left(\frac{2}{3}\right).$$

1.7.11 Observe the CBS Inequality in $\mathbb{R}^3$ given the vectors $\overrightarrow{x}=(x_1,x_2,x_3)$, $\overrightarrow{y}=(y_1,y_2,y_3)$ let θ be the angle between them. Then

$$\overrightarrow{x}\bullet\overrightarrow{y}=\left|\left|\overrightarrow{x}\right|\right|\left|\left|\overrightarrow{y}\right|\right|\cos\theta\implies|x_1y_1+x_2y_2+x_3y_3|\leq\sqrt{x_1^2+x_2^2+x_3^2}\sqrt{y_1^2+y_2^2+y_3^2}.$$

Now take $x_1=a^2, x_2=b^2, x_3=c^2$ and $y_1=y_2=y_3=1$. This gives (since squares are positive, we don't need the absolute values)

$$|x_1y_1+x_2y_2+x_3y_3|\leq\sqrt{x_1^2+x_2^2+x_3^2}\sqrt{y_1^2+y_2^2+y_3^2}\implies(a^2+b^2+c^2)^2\leq(a^4+b^4+c^4)(3),$$

which proves the claim at once.

1.7.12 First observe that

$$\begin{aligned}S(a,b,c)&=\sqrt{s(s-a)(s-b)(s-c)}\\&=\frac{1}{4}\sqrt{(a+b+c)(b+c-a)(c+a-b)(a+b-c)}\\&=\frac{1}{4}\sqrt{(a^2+b^2+c^2)^2-2(a^4+b^4+c^4)}\end{aligned}$$

Hence

$$\frac{S(a,b,c)}{a^2+b^2+c^2}=\frac{1}{4}\sqrt{1-2\frac{a^4+b^4+c^4}{(a^2+b^2+c^2)^2}},$$

and thus maximising f is equivalent to minimising $2\dfrac{a^4+b^4+c^4}{(a^2+b^2+c^2)^2}$. From problem 1.7.11,

$$\frac{a^4+b^4+c^4}{(a^2+b^2+c^2)^2}\geq\frac{1}{3},$$

which in turn gives

$$\frac{S(a,b,c)}{a^2+b^2+c^2}\leq\frac{1}{4}\sqrt{1-\frac{2}{3}}=\frac{1}{4\sqrt{3}}=\frac{\sqrt{3}}{12},$$

the desired maximum.

1.7.15 $x+y+z=1$. $\dfrac{1}{6}$.

1.7.16 Assume contrariwise that $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c}$ are three unit vectors in $\mathbb{R}^3$ such that the angle between any two of them is $>\dfrac{2\pi}{3}$. Then $\overrightarrow{a}\bullet\overrightarrow{b}<-\dfrac{1}{2}$, $\overrightarrow{b}\bullet\overrightarrow{c}<-\dfrac{1}{2}$, and $\overrightarrow{c}\bullet\overrightarrow{a}<-\dfrac{1}{2}$. Thus

$$\begin{aligned}\left|\left|\overrightarrow{a}+\overrightarrow{b}+\overrightarrow{c}\right|\right|^2&=\left|\left|\overrightarrow{a}\right|\right|^2+\left|\left|\overrightarrow{b}\right|\right|^2+\left|\left|\overrightarrow{c}\right|\right|^2\\&\quad+2\overrightarrow{a}\bullet\overrightarrow{b}+2\overrightarrow{b}\bullet\overrightarrow{c}+2\overrightarrow{c}\bullet\overrightarrow{a}\\&<1+1+1-1-1-1\\&=0,\end{aligned}$$

which is impossible, since a norm of vectors is always ≥ 0.

1.7.17 Let $\mathbf{proj}_{\mathbf{s}}^{\mathbf{t}} = \dfrac{\overrightarrow{\mathbf{t}}\bullet\overrightarrow{\mathbf{s}}}{(||s||)^2}\overrightarrow{\mathbf{s}}$ be the projection of $\overrightarrow{\mathbf{t}}$ over $\overrightarrow{\mathbf{s}} \neq \overrightarrow{\mathbf{0}}$. Let $\mathbf{x}_0$ be the point on the plane that is nearest to $\mathbf{b}$. Then $\overrightarrow{\mathbf{bx}_0} = \overrightarrow{\mathbf{x}_0} - \overrightarrow{\mathbf{b}}$ is orthogonal to the plane, and the distance we seek is

$$||\mathbf{proj}_{\mathbf{n}}^{\overrightarrow{\mathbf{r}_0}-\overrightarrow{\mathbf{b}}}|| = \left|\left|\frac{(\overrightarrow{\mathbf{r}_0}-\overrightarrow{\mathbf{b}})\bullet\overrightarrow{\mathbf{n}}}{||\overrightarrow{\mathbf{n}}||^2}\overrightarrow{\mathbf{n}}\right|\right| = \frac{|(\overrightarrow{\mathbf{r}_0}-\overrightarrow{\mathbf{b}})\bullet\overrightarrow{\mathbf{n}}|}{||\overrightarrow{\mathbf{n}}||}.$$

Since R_0 is on the plane, $\overrightarrow{\mathbf{r}_0}\bullet\overrightarrow{\mathbf{n}} = \overrightarrow{\mathbf{a}}\bullet\overrightarrow{\mathbf{n}}$, and so

$$||\mathbf{proj}_{\mathbf{n}}^{\overrightarrow{\mathbf{r}_0}-\overrightarrow{\mathbf{b}}}|| = \frac{|\overrightarrow{\mathbf{r}_0}\bullet\overrightarrow{\mathbf{n}} - \overrightarrow{\mathbf{b}}\bullet\overrightarrow{\mathbf{n}}|}{||\overrightarrow{\mathbf{n}}|||} = \frac{|\overrightarrow{\mathbf{a}}\bullet\overrightarrow{\mathbf{n}} - \overrightarrow{\mathbf{b}}\bullet\overrightarrow{\mathbf{n}}|}{||\overrightarrow{\mathbf{n}}||} = \frac{|(\overrightarrow{\mathbf{a}}-\overrightarrow{\mathbf{b}})\bullet\overrightarrow{\mathbf{n}}|}{||\overrightarrow{\mathbf{n}}||},$$

as we wanted to shew.

1.7.18 There are **7** vertices ($V_0 = (0,0,0)$, $V_1 = (11,0,0)$, $V_2 = (0,9,0)$, $V_3 = (0,0,8)$, $V_4 = (0,3,8)$, $V_5 = (9,0,2)$, $V_6 = (4,7,0)$) and **11** edges (V_0V_1, V_0V_2, V_0V_3, V_1V_5, V_1V_6, V_2V_4, V_2V_6, V_3V_4, V_3V_5, V_4V_5, and V_4V_6).

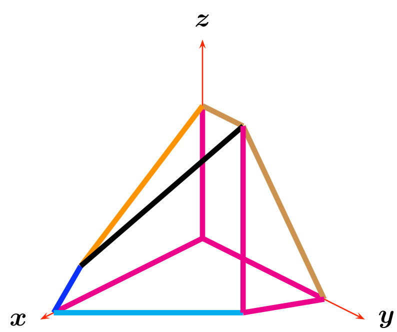

Figure A.11: Problem 1.7.18.

1.8.2 We have $\overrightarrow{\mathbf{x}} \times \overrightarrow{\mathbf{x}} = -\overrightarrow{\mathbf{x}} \times \overrightarrow{\mathbf{x}}$ by letting $\overrightarrow{\mathbf{y}} = \overrightarrow{\mathbf{x}}$ in **??**. Thus $2\overrightarrow{\mathbf{x}} \times \overrightarrow{\mathbf{x}} = \overrightarrow{\mathbf{0}}$ and hence $\overrightarrow{\mathbf{x}} \times \overrightarrow{\mathbf{x}} = \overrightarrow{\mathbf{0}}$.

1.8.3 One has

$$(\overrightarrow{\mathbf{b}} - \overrightarrow{\mathbf{a}}) \times (\overrightarrow{\mathbf{c}} - \overrightarrow{\mathbf{a}}) = \overrightarrow{\mathbf{0}} \implies \overrightarrow{\mathbf{a}} \times \overrightarrow{\mathbf{b}} + \overrightarrow{\mathbf{b}} \times \overrightarrow{\mathbf{c}} + \overrightarrow{\mathbf{c}} \times \overrightarrow{\mathbf{a}} = \overrightarrow{\mathbf{0}}$$

This gives

$$\overrightarrow{\mathbf{b}} \times \overrightarrow{\mathbf{c}} = -(\overrightarrow{\mathbf{a}} \times \overrightarrow{\mathbf{b}} + \overrightarrow{\mathbf{c}} \times \overrightarrow{\mathbf{a}}) = -(\overrightarrow{\mathbf{j}} + \overrightarrow{\mathbf{k}} + \overrightarrow{\mathbf{i}} - \overrightarrow{\mathbf{j}}) = -\overrightarrow{\mathbf{i}} - \overrightarrow{\mathbf{k}}.$$

1.8.4 The vector $\begin{bmatrix}1\\1\\1\end{bmatrix} \times \begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}-1\\1\\0\end{bmatrix}$ is normal to the plane. The plane has thus equation

$$\begin{bmatrix}x\\y+1\\z-2\end{bmatrix}\bullet\begin{bmatrix}-1\\1\\0\end{bmatrix} = 0 \implies -x+y+1 = 0 \implies x-y = 1,$$

as obtained before.

1.8.5 The vectors

$$\begin{bmatrix}a-(-a)\\0-1\\a-0\end{bmatrix} = \begin{bmatrix}2a\\-1\\a\end{bmatrix}$$

and

$$\begin{bmatrix}0-(-a)\\1-1\\2a-0\end{bmatrix} = \begin{bmatrix}a\\0\\2a\end{bmatrix}$$

lie on the plane. A vector normal to the plane is

$$\begin{bmatrix} 2a \\ -1 \\ a \end{bmatrix} \times \begin{bmatrix} a \\ 0 \\ 2a \end{bmatrix} = \begin{bmatrix} -2a \\ -3a^2 \\ a \end{bmatrix}.$$

The equation of the plane is thus given by

$$\begin{bmatrix} -2a \\ -3a^2 \\ a \end{bmatrix} \bullet \begin{bmatrix} x-a \\ y-0 \\ z-a \end{bmatrix} = 0,$$

that is,

$$2ax + 3a^2y - az = a^2.$$

1.8.6 Either of $\dfrac{\overrightarrow{v} \times \overrightarrow{w}}{||\overrightarrow{v} \times \overrightarrow{w}||}$ or $-\dfrac{\overrightarrow{v} \times \overrightarrow{w}}{||\overrightarrow{v} \times \overrightarrow{w}||}$ will do. Now

$$\begin{aligned} \overrightarrow{v} \times \overrightarrow{w} &= (-a\overrightarrow{j} + a\overrightarrow{k}) \times (\overrightarrow{i} + a\overrightarrow{j}) \\ &= -a(\overrightarrow{j} \times \overrightarrow{i}) - a^2(\overrightarrow{j} \times \overrightarrow{j}) + a(\overrightarrow{k} \times \overrightarrow{i}) + a^2(\overrightarrow{k} \times \overrightarrow{j}) \\ &= a\overrightarrow{k} + a\overrightarrow{j} - a^2\overrightarrow{i} \\ &= \begin{pmatrix} -a^2 \\ a \\ a \end{pmatrix}, \end{aligned}$$

and $||\overrightarrow{v} \times \overrightarrow{w}|| = \sqrt{a^4 + a^2 + a^2} = \sqrt{2a^2 + a^4}$. Hence we may take either

$$\frac{1}{\sqrt{2a^2 + a^4}} \begin{pmatrix} -a^2 \\ a \\ a \end{pmatrix}$$

or

$$-\frac{1}{\sqrt{2a^2 + a^4}} \begin{pmatrix} -a^2 \\ a \\ a \end{pmatrix}.$$

1.8.7 From Theorem **??** we have

$$\begin{aligned} \overrightarrow{a} \times (\overrightarrow{b} \times \overrightarrow{c}) &= (\overrightarrow{a} \bullet \overrightarrow{c})\overrightarrow{b} - (\overrightarrow{a} \bullet \overrightarrow{b})\overrightarrow{c}, \\ \overrightarrow{b} \times (\overrightarrow{c} \times \overrightarrow{a}) &= (\overrightarrow{b} \bullet \overrightarrow{a})\overrightarrow{c} - (\overrightarrow{b} \bullet \overrightarrow{c})\overrightarrow{a}, \\ \overrightarrow{c} \times (\overrightarrow{a} \times \overrightarrow{b}) &= (\overrightarrow{c} \bullet \overrightarrow{b})\overrightarrow{a} - (\overrightarrow{c} \bullet \overrightarrow{a})\overrightarrow{b}, \end{aligned}$$

and adding yields the result.

1.8.8 By Lagrange's Identity,

$$\begin{aligned} ||\overrightarrow{x} \times \overrightarrow{i}||^2 &= \left|\left|\overrightarrow{x}\right|\right|^2 \left|\left|\overrightarrow{i}\right|\right|^2 - (\overrightarrow{x} \bullet \overrightarrow{i})^2 = 1 - (\overrightarrow{x} \bullet \overrightarrow{i})^2, \\ ||\overrightarrow{x} \times \overrightarrow{k}||^2 &= \left|\left|\overrightarrow{x}\right|\right|^2 \left|\left|\overrightarrow{j}\right|\right|^2 - (\overrightarrow{x} \bullet \overrightarrow{j})^2 = 1 - (\overrightarrow{x} \bullet \overrightarrow{j})^2, \\ ||\overrightarrow{x} \times \overrightarrow{j}||^2 &= \left|\left|\overrightarrow{x}\right|\right|^2 \left|\left|\overrightarrow{k}\right|\right|^2 - (\overrightarrow{x} \bullet \overrightarrow{k})^2 = 1 - (\overrightarrow{x} \bullet \overrightarrow{k})^2, \end{aligned}$$

and since $(\overrightarrow{x} \bullet \overrightarrow{i})^2 + (\overrightarrow{x} \bullet \overrightarrow{j})^2 + (\overrightarrow{x} \bullet \overrightarrow{k})^2 = \left|\left|\overrightarrow{x}\right|\right|^2 = 1$, the desired sum equals $3 - 1 = 2$.

1.8.9

$$\overrightarrow{a} \times (\overrightarrow{x} \times \overrightarrow{b}) = \overrightarrow{b} \times (\overrightarrow{x} \times \overrightarrow{a}) \iff (\overrightarrow{a} \bullet \overrightarrow{b})\overrightarrow{x} - (\overrightarrow{a} \bullet \overrightarrow{x})\overrightarrow{b} = (\overrightarrow{b} \bullet \overrightarrow{a})\overrightarrow{x} - (\overrightarrow{b} \bullet \overrightarrow{x})\overrightarrow{a} \iff \overrightarrow{a} \bullet \overrightarrow{x} = \overrightarrow{b} \bullet \overrightarrow{x} = 0.$$

The answer is thus $\{\overrightarrow{x} : \overrightarrow{x} \in \mathbb{R}\overrightarrow{a} \times \overrightarrow{b}\}$.

1.8.12

$$\overrightarrow{x} = \frac{(\overrightarrow{a}\bullet\overrightarrow{b})\overrightarrow{a} + 6\overrightarrow{b} + 2\overrightarrow{a}\times\overrightarrow{c}}{12 + 2\left|\left|\overrightarrow{a}\right|\right|^2}$$

$$\overrightarrow{y} = \frac{(\overrightarrow{a}\bullet\overrightarrow{c})\overrightarrow{a} + 6\overrightarrow{c} + 3\overrightarrow{a}\times\overrightarrow{b}}{18 + 3\left|\left|\overrightarrow{a}\right|\right|^2}$$

1.8.13 First observe that

$$\overrightarrow{x}\bullet(\overrightarrow{y}\times\overrightarrow{z}) = (\overrightarrow{x}\times\overrightarrow{y})\bullet\overrightarrow{z}.$$

This is so because both sides give the volume of the parallelogram spanned by $\overrightarrow{x}$, $\overrightarrow{y}$, $\overrightarrow{z}$. Now, putting $\overrightarrow{x} = \overrightarrow{a}\times\overrightarrow{b}$, $\overrightarrow{y} = \overrightarrow{c}$ and $\overrightarrow{z} = \overrightarrow{d}$ we gather that

$$(\overrightarrow{a}\times\overrightarrow{b})\bullet(\overrightarrow{c}\times\overrightarrow{d}) \quad = \quad ((\overrightarrow{a}\times\overrightarrow{b})\times\overrightarrow{c})\bullet\overrightarrow{d}.$$

Now, again,

$$(\overrightarrow{a}\times\overrightarrow{b})\times\overrightarrow{c} = -\overrightarrow{c}\times(\overrightarrow{a}\times\overrightarrow{b}) = -((\overrightarrow{c}\bullet\overrightarrow{b})\overrightarrow{a} - (\overrightarrow{c}\bullet\overrightarrow{a})\overrightarrow{b}) = (\overrightarrow{c}\bullet\overrightarrow{a})\overrightarrow{b} - (\overrightarrow{c}\bullet\overrightarrow{b})\overrightarrow{a}.$$

This gives

$$((\overrightarrow{a}\times\overrightarrow{b})\times\overrightarrow{c})\bullet\overrightarrow{d} = ((\overrightarrow{c}\bullet\overrightarrow{a})\overrightarrow{b} - (\overrightarrow{c}\bullet\overrightarrow{b})\overrightarrow{a})\bullet\overrightarrow{d} = (\overrightarrow{c}\bullet\overrightarrow{a})(\overrightarrow{b}\bullet\overrightarrow{d}) - (\overrightarrow{c}\bullet\overrightarrow{b})(\overrightarrow{a}\bullet\overrightarrow{d}),$$

proving the identity.

1.8.14 By problem 1.8.13,

$$(\overrightarrow{x}\times\overrightarrow{y})\bullet(\overrightarrow{u}\times\overrightarrow{v}) = (\overrightarrow{x}\bullet\overrightarrow{u})\bullet(\overrightarrow{y}\bullet\overrightarrow{v}) - (\overrightarrow{x}\bullet\overrightarrow{v})\bullet(\overrightarrow{y}\bullet\overrightarrow{u}).$$

Using this three times:

$$\begin{array}{rcl}
(\overrightarrow{b}\times\overrightarrow{c})\bullet(\overrightarrow{a}\times\overrightarrow{d}) & = & (\overrightarrow{b}\bullet\overrightarrow{a})\bullet(\overrightarrow{c}\bullet\overrightarrow{d}) - (\overrightarrow{b}\bullet\overrightarrow{d})\bullet(\overrightarrow{c}\bullet\overrightarrow{d}) \\
(\overrightarrow{c}\times\overrightarrow{a})\bullet(\overrightarrow{b}\times\overrightarrow{d}) & = & (\overrightarrow{c}\bullet\overrightarrow{b})\bullet(\overrightarrow{a}\bullet\overrightarrow{d}) - (\overrightarrow{c}\bullet\overrightarrow{d})\bullet(\overrightarrow{a}\bullet\overrightarrow{b}) \\
(\overrightarrow{a}\times\overrightarrow{b})\bullet(\overrightarrow{c}\times\overrightarrow{d}) & = & (\overrightarrow{a}\bullet\overrightarrow{c})\bullet(\overrightarrow{b}\bullet\overrightarrow{d}) - (\overrightarrow{a}\bullet\overrightarrow{d})\bullet(\overrightarrow{b}\bullet\overrightarrow{c})
\end{array}$$

Adding these three equalities, and using the fact that the dot product is commutative, we see that all the terms on the dextral side cancel out and we obtain **0**, as required.

1.8.15

1. We have

$$\overrightarrow{CA} = \begin{bmatrix} 6 \\ 0 \\ -3 \end{bmatrix}, \quad \overrightarrow{CB} = \begin{bmatrix} 0 \\ 4 \\ -3 \end{bmatrix} \implies \overrightarrow{CA}\times\overrightarrow{CB} = (6\overrightarrow{i} - 3\overrightarrow{k})\times(4\overrightarrow{j} - 3\overrightarrow{k}) = 24\overrightarrow{k} + 18\overrightarrow{j} + 12\overrightarrow{i} = \begin{bmatrix} 12 \\ 18 \\ 24 \end{bmatrix}.$$

2. We have

$$\left|\left|\overrightarrow{CA}\times\overrightarrow{CB}\right|\right| = \sqrt{12^2 + 18^2 + 24^2} = \sqrt{1044} = 6\sqrt{29}.$$

3. The desired line has equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} + t\begin{bmatrix} 6 \\ 0 \\ -3 \end{bmatrix} \implies x = 6t, \quad y = 0, \quad z = 3 - 3t.$$

4. The desired line has equation

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} + s\begin{bmatrix} 0 \\ 0 \\ -3 \end{bmatrix} \implies x = 3, \quad y = 0, \quad z = -3s.$$

5. From the preceding items, the line L_{CA} is $x = 6t,\ y = 0,\ z = 3 - 3t$ and the line L_{DE} is $x = 3,\ y = 0,\ z = -3s$. If the line intersect then $6t = 3,\ 0 = 0,\ 3 - 3t = -3s$ gives $t = \frac{1}{2}$ and $s = -\frac{1}{2}$. The point of intersection is thus $\left(3, 0, \frac{3}{2}\right)$. item The area is
$$\frac{1}{2}\left|\left|\overrightarrow{CA} \times \overrightarrow{CB}\right|\right| = \frac{1}{2} \cdot 6\sqrt{29} = 3\sqrt{29}.$$

6. Observe that
$$P = \begin{pmatrix} 3 \\ 0 \\ z \end{pmatrix}, \quad Q = \begin{pmatrix} 3 \\ y \\ 0 \end{pmatrix}, \quad R = \begin{pmatrix} x \\ 3 \\ 0 \end{pmatrix}, \quad S = \begin{pmatrix} 0 \\ 3 \\ z \end{pmatrix}.$$
Since all this points lie on the plane $2x + 3y + 4z = 12$, we find
$$2(3) + 3(0) + 4z = 12 \implies P = \begin{pmatrix} 3 \\ 0 \\ \frac{3}{2} \end{pmatrix},$$
$$2(3) + 3y + 4(0) = 12 \implies Q = \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix},$$
$$2x + 3(3) + 4(0) = 12 \implies R = \begin{pmatrix} \frac{3}{2} \\ 3 \\ 0 \end{pmatrix},$$
$$2(0) + 3(3) + 4z = 12 \implies S = \begin{pmatrix} 0 \\ 3 \\ \frac{3}{4} \end{pmatrix}.$$

7. A possible way is to decompose the pentagon into three triangles, say $\triangle CPQ$, $\triangle CQR$ and $\triangle CRS$ and find their areas. Another way would be to subtract from the area of $\triangle ABC$ the areas of $\triangle APQ$ and $\triangle RSB$. I will follow the second approach. Let $[\triangle APQ]$, $[\triangle RSB]$ denote the areas of $\triangle APQ$ and $\triangle RSB$ respectively. Then
$$[\triangle APQ] = \frac{1}{2}\left|\left|\overrightarrow{PA} \times \overrightarrow{PQ}\right|\right| = \frac{1}{2}\left|\left|\begin{bmatrix} 3 \\ 0 \\ \frac{3}{2} \end{bmatrix} \times \begin{bmatrix} 0 \\ 2 \\ -\frac{3}{2} \end{bmatrix}\right|\right| = \frac{1}{2}\left|\left|\begin{bmatrix} -3 \\ \frac{9}{2} \\ 6 \end{bmatrix}\right|\right| = \frac{1}{2}\sqrt{9 + \frac{81}{4} + 36} = \frac{3}{4}\sqrt{29},$$
$$[\triangle RSB] = \frac{1}{2}\left|\left|\overrightarrow{SR} \times \overrightarrow{SB}\right|\right| = \frac{1}{2}\left|\left|\begin{bmatrix} \frac{3}{2} \\ 0 \\ -\frac{3}{4} \end{bmatrix} \times \begin{bmatrix} 0 \\ 1 \\ -\frac{3}{4} \end{bmatrix}\right|\right| = \frac{1}{2}\left|\left|\begin{bmatrix} \frac{3}{4} \\ \frac{9}{8} \\ \frac{3}{2} \end{bmatrix}\right|\right| = \frac{1}{2}\sqrt{\frac{9}{16} + \frac{81}{64} + \frac{9}{4}} = \frac{3}{16}\sqrt{29}.$$
Hence the area of the pentagon is
$$3\sqrt{29} - \frac{3}{4}\sqrt{29} - \frac{3}{16}\sqrt{29} = \frac{33}{16}\sqrt{29}.$$

1.9.2 The assertion is trivial for $n = 1$. Assume its truth for $n - 1$, that is, assume $A^{n-1} = 3^{n-2}A$. Observe that
$$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 & 3 \\ 3 & 3 & 3 \\ 3 & 3 & 3 \end{bmatrix} = 3A.$$
Now
$$A^n = AA^{n-1} = A(3^{n-2}A) = 3^{n-2}A^2 = 3^{n-2}3A = 3^{n-1}A,$$
and so the assertion is proved by induction.

1.9.3 First, we will prove that

$$A^2 = \begin{bmatrix} 1 & 2 & 3 & 4 & \dots & n-1 & n \\ 0 & 1 & 2 & 3 & \dots & n-2 & n-1 \\ 0 & 0 & 1 & 2 & \dots & n-3 & n-2 \\ \dots & \dots & \vdots & \vdots & \vdots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}.$$

Observe that $A = [a_{ij}]$, where $a_{ij} = 1$ for $i \leq j$ and $a_{ij} = 0$ for $i > j$.

Put $A^2 = [b_{ij}]$. Assume first that $i \leq j$. Then

$$b_{ij} = \sum_{k=1}^{n} a_{ik}a_{kj} = \sum_{k=i}^{j} 1 = j - i + 1.$$

Assume now that $i > j$. Then

$$b_{ij} = \sum_{k=1}^{n} a_{ik}a_{kj} = \sum_{k=1}^{n} 0 = 0,$$

proving the first statement. Now, we will prove that

$$A^3 = \begin{bmatrix} 1 & 3 & 6 & 10 & \dots & \frac{(n-1)n}{2} & \frac{n(n+1)}{2} \\ 0 & 1 & 3 & 6 & \dots & \frac{(n-2)(n-1)}{2} & \frac{(n-1)n}{2} \\ 0 & 0 & 1 & 3 & \dots & \frac{(n-3)(n-2)}{2} & \frac{(n-2)(n-1)}{2} \\ \dots & \dots & \vdots & \vdots & \vdots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 0 & 1 \end{bmatrix}.$$

For the second part, you need to know how to sum arithmetic progressions. In our case, we need to know how to sum (assume $i \leq j$),

$$S_1 = \sum_{k=i}^{j} a, \quad S_2 = \sum_{k=i}^{j} k.$$

The first sum is trivial: there are $j - i + 1$ integers in the interval $[i; j]$, and hence

$$S_1 = \sum_{k=i}^{j} a = S_1 = a \sum_{k=i}^{j} 1 = a(j - i + 1).$$

For the second sum, we use Gauß trick: summing the sum forwards is the same as summing the sum backwards, and so, adding the first two rows below,

$$\begin{array}{lllllllllllll} S_2 & = & i & + & i+1 & + & i+2 & + & \cdots & + & j-1 & + & j \\ S_2 & = & j & + & j-1 & + & j-2 & + & \cdots & + & i-1 & + & i \\ \hline 2S_2 & = & (i+j) & + & (i+j) & + & (i+j) & + & \cdots & + & (i+j) & + & (i+j) \\ \hline 2S_2 & = & (i+j)(j-i+1) & & & & & & \cdots & & & & \end{array},$$

which gives $S_2 = \dfrac{(i+j)(j-i+1)}{2}$.

Put now $A^3 = [c_{ij}]$. Assume first that $i \leq j$. Since $A^3 = A^2A$,

$$\begin{aligned} c_{ij} &= \sum_{k=1}^{n} b_{ik}a_{kj} \\ &= \sum_{k=i}^{j} (k - i + 1) \\ &= \sum_{k=i}^{j} k - \sum_{k=i}^{j} i + \sum_{k=i}^{j} 1 \\ &= \frac{(j+i)(j-i+1)}{2} - i(j-i+1) + (j-i+1) \\ &= \frac{(j-i+1)(j-i+2)}{2}. \end{aligned}$$

Assume now that $i > j$. Then

$$c_{ij} = \sum_{k=1}^{n} b_{ik}a_{kj} = \sum_{k=1}^{n} 0 = 0.$$

This finishes the proof.

1.9.4 For the first part, observe that

$$\begin{aligned}
m(a)m(b) &= \begin{bmatrix} 1 & 0 & a \\ -a & 1 & -\frac{a^2}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & b \\ -b & 1 & -\frac{b^2}{2} \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & a+b \\ -a-b & 1 & -\frac{a^2}{2} - \frac{b^2}{2} + ab \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & a+b \\ -(a+b) & 1 & -\frac{(a+b)^2}{2} \\ 0 & 0 & 1 \end{bmatrix} \\
&= m(a+b)
\end{aligned}$$

For the second part, observe that using the preceding part of the problem,

$$m(a)m(-a) = m(a-a) = m(0) = \begin{bmatrix} 1 & 0 & 0 \\ -0 & 1 & -\frac{0^2}{2} \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}_3,$$

giving the result.

1.11.1 $\frac{a}{\sqrt{2}}, \frac{\pi}{4}$.

1.11.3 $\frac{a}{\sqrt{3}}$

1.12.1 Consider a right triangle $\triangle ABC$ rectangle at A with legs of length $CA = h$ and $AB = r$, as in figure A.12. The cone is generated when the triangle rotates about CA. The gyrating curve is the hypotenuse, whose centroid is its centre. The length of the generating curve is thus $\sqrt{r^2 + h^2}$ and the length of curve described by the centre of gravity is $2\pi\left(\frac{r}{2}\right) = \pi r$. The lateral area is thus $\pi r\sqrt{r^2 + h^2}$.

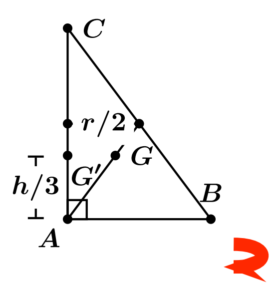

Figure A.12: Generating a cone.

To find the volume, we gyrate the whole right triangle, whose area is $\frac{rh}{2}$. We need to find the centroid of the triangle. But from example 17, we know that the centroid G of the triangle is is two thirds of the way from A to the midpoint of BC. If G' is the perpendicular projection of G onto $[CA]$, then this means that G' is at a vertical height of $\frac{h}{2}\cdot\frac{2}{3}=\frac{h}{3}$. By similar triangles $\frac{GG'}{r}=\frac{h/3}{h} \implies GG'=\frac{r}{3}$. Hence, the length of the curve described by the centre of gravity of the triangle is $\frac{2}{3}\pi r$. The volume of the cone is thus $\frac{2}{3}\pi r\cdot\frac{rh}{2}=\frac{\pi}{3}r^2h$.

1.14.1 Find a vector $\overrightarrow{a}$ mutually perpendicular to $\overrightarrow{V_1V_2}$ and $\overrightarrow{V_1V_3}$ and another vector and a vector $\overrightarrow{b}$ mutually perpendicular to $\overrightarrow{V_1V_3}$ and $\overrightarrow{V_1V_4}$. Then shew that $\cos\theta=\frac{1}{3}$, where θ is the angle between $\overrightarrow{a}$ and $\overrightarrow{b}$.

1.15.1 Let $\begin{pmatrix}x\\y\\z\end{pmatrix}$ be a point on S. If this point were on the xz plane, it would be on the ellipse, and its distance to the axis of rotation would be $|x|=\frac{1}{2}\sqrt{1-z^2}$. Anywhere else, the distance from $\begin{pmatrix}x\\y\\z\end{pmatrix}$ to the z-axis is the distance of this point to the point $\begin{pmatrix}0\\0\\z\end{pmatrix}$: $\sqrt{x^2+y^2}$. This distance is the same as the length of the segment on the xz-plane going from the z-axis. We thus have

$$\sqrt{x^2+y^2}=\frac{1}{2}\sqrt{1-z^2},$$

or

$$4x^2+4y^2+z^2=1.$$

1.15.2 Let $\begin{pmatrix}x\\y\\z\end{pmatrix}$ be a point on S. If this point were on the xy plane, it would be on the line, and its distance to the axis of rotation would be $|x|=\frac{1}{3}|1-4y|$. Anywhere else, the distance of $\begin{pmatrix}x\\y\\z\end{pmatrix}$ to the axis of rotation is the same as the distance of $\begin{pmatrix}x\\y\\z\end{pmatrix}$ to $\begin{pmatrix}0\\y\\0\end{pmatrix}$, that is $\sqrt{x^2+z^2}$. We must have

$$\sqrt{x^2+z^2}=\frac{1}{3}|1-4y|,$$

which is to say

$$9x^2+9z^2-16y^2+8y-1=0.$$

1.15.3 A spiral staircase.

1.15.4 A spiral staircase.

1.15.6 The planes $A : x+z=0$ and $B : y=0$ are secant. The surface has equation of the form $f(A,B)=e^{A^2+B^2}-A=0$, and it is thus a cylinder. The directrix has direction $\overrightarrow{i}-\overrightarrow{k}$.

1.15.7 Rearranging,

$$(x^2+y^2+z^2)^2-\frac{1}{2}((x+y+z)^2-(x^2+y^2+z^2))-1=0,$$

and so we may take $A : x+y+z=0, \Sigma : x^2+y^2+z^2=0$, shewing that the surface is of revolution. Its axis is the line in the direction $\overrightarrow{i}+\overrightarrow{j}+\overrightarrow{k}$.

1.15.8 Considering the planes $A : x - y = 0, B : y - z = 0$, the equation takes the form

$$f(A, B) = \frac{1}{A} + \frac{1}{B} - \frac{1}{A + B} - 1 = 0,$$

thus the equation represents a cylinder. To find its directrix, we find the intersection of the planes $x = y$ and $y = z$. This gives $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. The direction vector is thus $\overrightarrow{i} + \overrightarrow{j} + \overrightarrow{k}$.

1.15.9 Rearranging,

$$(x + y + z)^2 - (x^2 + y^2 + z^2) + 2(x + y + z) + 2 = 0,$$

so we may take $A : x + y + z = 0, \Sigma : x^2 + y^2 + z^2 = 0$ as our plane and sphere. The axis of revolution is then in the direction of $\overrightarrow{i} + \overrightarrow{j} + \overrightarrow{k}$.

1.15.10 After rearranging, we obtain

$$(z - 1)^2 - xy = 0,$$

or

$$-\frac{x}{z - 1}\frac{y}{z - 1} + 1 = 0.$$

Considering the planes

$$A : x = 0, \ B : y = 0, \ \ C : z = 1,$$

we see that our surface is a cone, with apex at $(0, 0, 1)$.

1.15.11 The largest circle has radius b. Parallel cross sections of the ellipsoid are similar ellipses, hence we may increase the size of these by moving towards the centre of the ellipse. Every plane through the origin which makes a circular cross section must intersect the yz-plane, and the diameter of any such cross section must be a diameter of the ellipse $x = 0, \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. Therefore, the radius of the circle is at most b. Arguing similarly on the xy-plane shews that the radius of the circle is at least b. To shew that circular cross section of radius b actually exist, one may verify that the two planes given by $a^2(b^2 - c^2)z^2 = c^2(a^2 - b^2)x^2$ give circular cross sections of radius b.

1.15.12 Any hyperboloid oriented like the one on the figure has an equation of the form

$$\frac{z^2}{c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1.$$

When $z = 0$ we must have

$$4x^2 + y^2 = 1 \implies a = \frac{1}{2}, \ b = 1.$$

Thus

$$\frac{z^2}{c^2} = 4x^2 + y^2 - 1.$$

Hence, letting $z = \pm 2$,

$$\frac{4}{c^2} = 4x^2 + y^2 - 1 \implies \frac{1}{c^2} = x^2 + \frac{y^2}{4} - \frac{1}{4} = 1 - \frac{1}{4} = \frac{3}{4},$$

since at $z = \pm 2$, $x^2 + \frac{y^2}{4} = 1$. The equation is thus

$$\frac{3z^2}{4} = 4x^2 + y^2 - 1.$$

1.16.1 The arc length element is

$$\sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2} = \sqrt{4t^2 + 36t + 81}\ \mathrm{d}t = (2t + 9)\ \mathrm{d}t.$$

We need $t = 1$ to $t = 4$. The desired length is

$$\int_1^4 (2t + 9)\ \mathrm{d}t = (t^2 + 9t)\Big|_1^4 = (16 + 36) - (1 + 9) = 42.$$

1.16.2 Observe that $z = 1+x^2+y^2$ is a paraboloid opening up, with vertex (lowest point) at $(0,0,1)$. On the other hand, $z = 3 - x^2 - y^2$ is another paraboloid opening down, with highest point at $(0,0,3)$. Adding the equations we obtain

$$2z = z + z = (1 + x^2 + y^2) + (3 - x^2 - y^2) = 4 \implies 2z = 4 \implies z = 2,$$

so they intersect at the plane $z = 2$. Subtracting the equations,

$$(1 + x^2 + y^2) - (3 - x^2 - y^2) = z - z = 0 \implies 2x^2 + 2y^2 - 2 = 0 \implies x^2 + y^2 = 1,$$

so they intersect at the circle $x^2 + y^2 = 1, z = 2$. Since they meet at the circle $x^2 + y^2 = 1$, we may parametrise this circle as $x = \cos t, y = \sin t$, $t \in [0; 2\pi]$, $z = 2$.

1.16.3 Let $\overrightarrow{r}(t)$ lie on the plane $ax + by + cz = d$. Then we must have

$$a\frac{t^4}{1+t^2} + b\frac{t^3}{1+t^2} + c\frac{t^2}{1+t^2} = d \implies (at^4 + bt^3 + ct^2) = d(1 + t^2) \implies at^4 + bt^3 + (c-d)t^2 - d = 0,$$

which means that if $\overrightarrow{r}(t)$ is on the plane $ax + by + cz = d$, then t must satisfy the quartic polynomial $p(t) = at^4 + bt^3 + (c-d)t^2 - d = 0$. Hence, the t_k are coplanar if and only if they are roots of $p(t)$. Since the coefficient of t in this polynomial is 0, then the sum of the roots of $p(t)$ taken three at a time is 0, that is,

$$t_1t_2t_3 + t_1t_2t_4 + t_1t_3t_4 + t_2t_3t_4 = 0 \implies \frac{t_1t_2t_3 + t_1t_2t_4 + t_1t_3t_4 + t_2t_3t_4}{t_1t_2t_3t_4} = 0 \implies \frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4} = 0,$$

as required.

1.16.4 Observe that in this problem you are *only parametrising the ellipsoid!* The tricky part is to figure out the bounds in your parameters so that *only* the part above the plane $x + y + z = 0$ is described. A common parametrisation found was:

$$x = \cos\theta\sin\phi, \quad y = 3\sin\theta\sin\phi, \quad z = \cos\phi.$$

The projection of the plane $x + y + z = 0$ onto the xy-plane is the line $y = -x$. To be "above" this line, the angle θ, measured from the positive x-axis needs to be in the interval $-\frac{\pi}{4} \le \theta \le \frac{3\pi}{4}$. Since $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is normal to the plane $x + y + z = 0$, the plane makes an angle of $\frac{\pi}{4}$ with the z-axis. Recall that ϕ is measured from $\phi = 0$ (positive z-axis) to $\phi = \pi$ (negative z-axis). Hence to be above the plane we need $0 \le \phi \le \frac{3\pi}{4}$.

1.16.6

1. $\begin{bmatrix} -2a \\ a^2 - 1 \\ a^2 + 1 \end{bmatrix}$
2. $x^2 + y^2 = c^2 + 1$
3. $\pi \int_0^1 (c^2 + 1)\mathrm{d}c = \frac{4\pi}{3}$

1.17.1

1. Put $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^{x-1} - x$. Clearly $f(1) = e^0 - 1 = 0$. Now,

$$f'(x) = e^{x-1} - 1,$$

$$f''(x) = e^{x-1}.$$

If $f'(x) = 0$ then $e^{x-1} = 1$ implying that $x = 1$. Thus f has a single minimum point at $x = 1$. Thus for all real numbers x

$$0 = f(1) \le f(x) = e^{x-1} - x,$$

which gives the desired result.

2. Easy Algebra!
3. Easy Algebra!

4. By the preceding results, we have
$$\begin{aligned} A_1 &\leq \exp(A_1 - 1), \\ A_2 &\leq \exp(A_2 - 1), \\ &\vdots \\ A_n &\leq \exp(A_n - 1). \end{aligned}$$
Since all the quantities involved are non-negative, we may multiply all these inequalities together, to obtain,
$$A_1 A_2 \cdots A_n \leq \exp(A_1 + A_2 + \cdots + A_n - n).$$
In view of the observations above, the preceding inequality is equivalent to
$$\frac{n^n G_n}{(a_1 + a_2 + \cdots + a_n)^n} \leq \exp(n - n) = e^0 = 1.$$
We deduce that
$$G_n \leq \left(\frac{a_1 + a_2 + \cdots + a_n}{n}\right)^n,$$
which is equivalent to
$$(a_1 a_2 \cdots a_n)^{1/n} \leq \frac{a_1 + a_2 + \cdots + a_n}{n}.$$
Now, for equality to occur, we need each of the inequalities $A_k \leq \exp(A_k - 1)$ to hold. This occurs, in view of the preceding lemma, if and only if $A_k = 1, \ \forall k$, which translates into $a_1 = a_2 = \ldots = a_n$. This completes the proof.

1.17.2 By CBS,
$$\begin{aligned} (x_1 + x_2 + \ldots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}\right) &\geq \left(\sum_{i=1}^n \sqrt{x_i}\frac{1}{\sqrt{x_i}}\right)^2 \\ &= n^2. \end{aligned}$$

1.17.3 By CBS,
$$(a + b + c + d)^2 \leq (1 + 1 + 1 + 1)\left(a^2 + b^2 + c^2 + d^2\right) = 4\left(a^2 + b^2 + c^2 + d^2\right).$$
Hence,
$$(8 - e)^2 \leq 4\left(16 - e^2\right) \iff e\,(5e - 16) \leq 0 \iff 0 \leq e \leq \frac{16}{5}.$$
The maximum value $e = \dfrac{16}{5}$ is reached when $a = b = c = d = \dfrac{6}{5}$.

1.17.4 Observe that $96 \cdot 216 = 144^2$ and by CBS,
$$\sum_{k=1}^n a_k^2 \leq \left(\sum_{k=1}^n a_k^3\right)\left(\sum_{k=1}^n a_k\right).$$
As there is equality,
$$(a_1, a_2, \ldots, a_n) = t(a_1^3, a_2^3, \ldots, a_n^3)$$
for some real number t. Hence $a_1 = a_2 = \ldots = a_n = a$, from where $na = 96$, $na^2 = 144$ gives $a = \dfrac{3}{2}$ y $n = 32$.

1.17.5 Applying the AM-GM inequality, for $1, 2, \ldots, n$:
$$n!^{1/n} = (1 \cdot 2 \cdots n)^{1/n} < \frac{1 + 2 + \cdots + n}{n} = \frac{n+1}{2},$$
with strict inequality for $n > 1$.

1.17.6 If $x \in [-a; a]$, then $a + x \geq 0$ and $a - x \geq 0$, and thus we may use AM-GM with $n = 8$, $a_1 = a_2 = \cdots = a_5 = \dfrac{a+x}{5}$ and $a_6 = a_7 = a_8 = \dfrac{a-x}{3}$. We deduce that
$$\left(\frac{a+x}{5}\right)^5\left(\frac{a-x}{3}\right)^3 \leq \left(\frac{5\left(\frac{a+x}{5}\right) + 3\left(\frac{a-x}{3}\right)}{8}\right)^8 = \left(\frac{a}{4}\right)^8,$$
from where
$$f(x) \leq \frac{5^5 3^3 a^8}{4^8},$$
with equality if and only if $\dfrac{a+x}{5} = \dfrac{a-x}{3}$.

1.17.7 Applying AM-GM to the set of $n+1$ numbers

$$1, 1+\frac{1}{n}, 1+\frac{1}{n}, \ldots, 1+\frac{1}{n},$$

has arithmetic mean

$$1+\frac{1}{n+1}$$

and geometric mean

$$\left(1+\frac{1}{n}\right)^{n/(n+1)}.$$

Therefore,

$$1+\frac{1}{n+1} > \left(1+\frac{1}{n}\right)^{n/(n+1)},$$

that is

$$\left(1+\frac{1}{n+1}\right)^{n+1} > \left(1+\frac{1}{n}\right)^{n},$$

which means

$$x_{n+1} > x_n,$$

giving the assertion.

2.1.2 Since polynomials are continuous functions and the image of a connected set is connected for a continuous function, the image must be an interval of some sort. If the image were a finite interval, then $f(x, kx)$ would be bounded for every constant k, and so the image would just be the point $f(0, 0)$. The possibilities are thus

1. a single point (take for example, $p(x, y) = 0$),
2. a semi-infinite interval with an endpoint (take for example $p(x, y) = x^2$ whose image is $[0; +\infty[$),
3. a semi-infinite interval with no endpoint (take for example $p(x, y) = (xy-1)^2 + x^2$ whose image is $]0; +\infty[$),
4. all real numbers (take for example $p(x, y) = x$).

2.3.4 0

2.3.5 2

2.3.6 $c = 0$.

2.3.7 0

2.3.10 By AM-GM,

$$\frac{x^2y^2z^2}{x^2+y^2+z^2} \leq \frac{(x^2+y^2+z^2)^3}{27(x^2+y^2+z^2)} = \frac{(x^2+y^2+z^2)^2}{27} \to 0$$

as $(x, y, z) \to (0, 0, 0)$.

2.4.1 We have

$$\begin{aligned} F(\overrightarrow{x}+\overrightarrow{h}) - F(\overrightarrow{x}) &= (\overrightarrow{x}+\overrightarrow{h}) \times L(\overrightarrow{x}+\overrightarrow{h}) - \overrightarrow{x} \times L(\overrightarrow{x}) \\ &= (\overrightarrow{x}+\overrightarrow{h}) \times (L(\overrightarrow{x}) + L(\overrightarrow{h})) - \overrightarrow{x} \times L(\overrightarrow{x}) \\ &= \overrightarrow{x} \times L(\overrightarrow{h}) + \overrightarrow{h} \times L(\overrightarrow{x}) + \overrightarrow{h} \times L(\overrightarrow{h}) \end{aligned}$$

Now, we will prove that $||\overrightarrow{h} \times L(\overrightarrow{h})|| = o\left(\left|\left|\overrightarrow{h}\right|\right|\right)$ as $\overrightarrow{h} \to \overrightarrow{0}$. For let

$$\overrightarrow{h} = \sum_{k=1}^{n} h_k \overrightarrow{e}_k,$$

where the $\overrightarrow{e}_k$ are the standard basis for $\mathbb{R}^n$. Then

$$L(\overrightarrow{h}) = \sum_{k=1}^{n} h_k L(\overrightarrow{e}_k),$$

and hence by the triangle inequality, and by the Cauchy-Bunyakovsky-Schwarz inequality,

$$\begin{aligned}
||L(\overrightarrow{h})|| &\leq \sum_{k=1}^{n} |h_k|||L(\overrightarrow{e}_k)|| \\
&\leq \left(\sum_{k=1}^{n} |h_k|^2\right)^{1/2} \left(\sum_{k=1}^{n} ||L(\overrightarrow{e}_k)||^2\right)^{1/2} \\
&= \left|\left|\overrightarrow{h}\right|\right| (\sum_{k=1}^{n} ||L(\overrightarrow{e}_k)||^2)^{1/2},
\end{aligned}$$

whence, again by the Cauchy-Bunyakovsky-Schwarz Inequality,

$$||\overrightarrow{h} \times L(\overrightarrow{h})|| \leq ||\overrightarrow{h}||||L(\overrightarrow{h})| \leq ||\overrightarrow{h}||^2|||L(\overrightarrow{e}_k)||^2)^{1/2} = o\left(||\overrightarrow{h}||\right),$$

as it was to be shewn.

2.4.2 Assume that $\overrightarrow{x} \neq \overrightarrow{0}$. We use the fact that $(1+t)^{1/2} = 1 + \frac{t}{2} + o(t)$ as $t \to 0$. We have

$$\begin{aligned}
f(\overrightarrow{x} + \overrightarrow{h}) - f(\overrightarrow{x}) &= ||\overrightarrow{x} + \overrightarrow{h}|| - \left|\left|\overrightarrow{x}\right|\right| \\
&= \sqrt{(\overrightarrow{x} + \overrightarrow{h})\bullet(\overrightarrow{x} + \overrightarrow{h})} - \left|\left|\overrightarrow{x}\right|\right| \\
&= \sqrt{\left|\left|\overrightarrow{x}\right|\right|^2 + 2\overrightarrow{x}\bullet\overrightarrow{h} + \left|\left|\overrightarrow{h}\right|\right|^2} - \left|\left|\overrightarrow{x}\right|\right| \\
&= \frac{2\overrightarrow{x}\bullet\overrightarrow{h} + \left|\left|\overrightarrow{h}\right|\right|^2}{\sqrt{\left|\left|\overrightarrow{x}\right|\right|^2 + 2\overrightarrow{x}\bullet\overrightarrow{h} + \left|\left|\overrightarrow{h}\right|\right|^2} + \left|\left|\overrightarrow{x}\right|\right|}.
\end{aligned}$$

As $\overrightarrow{h} \to \overrightarrow{0}$,

$$\sqrt{\left|\left|\overrightarrow{x}\right|\right|^2 + 2\overrightarrow{x}\bullet\overrightarrow{h} + \left|\left|\overrightarrow{h}\right|\right|^2} + \left|\left|\overrightarrow{x}\right|\right| \to 2\left|\left|\overrightarrow{x}\right|\right|.$$

Since $\left|\left|\overrightarrow{h}\right|\right|^2 = o\left(\left|\left|\overrightarrow{h}\right|\right|\right)$ as $\overrightarrow{h} \to \overrightarrow{0}$, we have

$$\frac{2\overrightarrow{x}\bullet\overrightarrow{h} + \left|\left|\overrightarrow{h}\right|\right|^2}{\sqrt{\left|\left|\overrightarrow{x}\right|\right|^2 + 2\overrightarrow{x}\bullet\overrightarrow{h} + \left|\left|\overrightarrow{h}\right|\right|^2} + \left|\left|\overrightarrow{x}\right|\right|} \to \frac{\overrightarrow{x}\bullet\overrightarrow{h}}{\left|\left|\overrightarrow{h}\right|\right|} + o\left(\left|\left|\overrightarrow{h}\right|\right|\right),$$

proving the first assertion.

To prove the second assertion, assume that there is a linear transformation $\mathscr{D}_0(f) = L$, $L : \mathbb{R}^n \to \mathbb{R}$ such that

$$||f(\overrightarrow{0} + \overrightarrow{h}) - f(\overrightarrow{0}) - L(\overrightarrow{h})|| = o\left(\left|\left|\overrightarrow{h}\right|\right|\right),$$

as $\left|\left|\overrightarrow{h}\right|\right| \to 0$. Recall that by theorem **??**, $L(\overrightarrow{0}) = \overrightarrow{0}$, and so by example 169, $\mathscr{D}_0(L)(\overrightarrow{0}) = L(\overrightarrow{0}) = \overrightarrow{0}$. This implies that $\frac{L(\overrightarrow{h})}{\left|\left|\overrightarrow{h}\right|\right|} \to \mathscr{D}_0(L)(\overrightarrow{0}) = \overrightarrow{0}$, as $\left|\left|\overrightarrow{h}\right|\right| \to 0$. Since $f(\overrightarrow{0}) = ||0|| = 0, f(\overrightarrow{h}) = \left|\left|\overrightarrow{h}\right|\right|$ this would imply that

$$\left|\left|\left|\left|\overrightarrow{h}\right|\right| - L(\overrightarrow{h})\right|\right| = o\left(\left|\left|\overrightarrow{h}\right|\right|\right),$$

or

$$\left|\left|1 - \frac{L(\overrightarrow{h})}{\left|\left|\overrightarrow{h}\right|\right|}\right|\right| = o(1).$$

But the sinistral side $\to 1$ as $\overrightarrow{h} \to \overrightarrow{0}$, and the dextral side $\to 0$ as $\overrightarrow{h} \to \overrightarrow{0}$. This is a contradiction, and so, such linear transformation L does not exist at the point $\overrightarrow{0}$.

2.5.2 Observe that

$$f(x, y) = \begin{cases} x & \text{if } x \leq y^2 \\ y^2 & \text{if } x > y^2 \end{cases}$$

Hence

$$\frac{\partial}{\partial x} f(x, y) = \begin{cases} 1 & \text{if } x > y^2 \\ 0 & \text{if } x > y^2 \end{cases}$$

and

$$\frac{\partial}{\partial y} f(x, y) = \begin{cases} 0 & \text{if } x > y^2 \\ 2y & \text{if } x > y^2 \end{cases}$$

2.5.3 Observe that

$$g(1, 0, 1) = \begin{bmatrix} 3 \\ 0 \end{bmatrix}, \qquad f'(x, y) = \begin{bmatrix} y^2 & 2xy \\ 2xy & x^2 \end{bmatrix}, \qquad g'(x, y) = \begin{bmatrix} 1 & -1 & 2 \\ y & x & 0 \end{bmatrix},$$

and hence

$$g'(1, 0, 1) = \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \end{bmatrix}, \qquad f'(g(1, 0, 1)) = f'(3, 0) = \begin{bmatrix} 0 & 0 \\ 0 & 9 \end{bmatrix}.$$

This gives, via the Chain-Rule,

$$(f \circ g)'(1, 0, 1) = f'(g(1, 0, 1))g'(1, 0, 1) = \begin{bmatrix} 0 & 0 \\ 0 & 9 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 9 & 0 \end{bmatrix}.$$

The composition $g \circ f$ is undefined. For, the output of f is $\mathbb{R}^2$, but the input of g is in $\mathbb{R}^3$.

2.5.4 Since $f(0, 1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, the Chain Rule gives

$$(g \circ f)'(0, 1) = (g'(f(0, 1)))(f'(0, 1)) = (g'(0, 1))(f'(0, 1)) = \begin{bmatrix} 1 & -1 \\ 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 0 \\ 2 & 1 \end{bmatrix}$$

2.5.5 We have

$$\frac{\partial f}{\partial x}(x, y, z) = 2xyg(x^2y),$$

and

$$\frac{\partial f}{\partial y}(x, y, z) = x^2g(x^2y).$$

2.5.6 Differentiating both sides with respect to the parameter a, the integral is $\dfrac{1}{2a^3} \arctan \dfrac{b}{a} + \dfrac{b}{2a^2(a^2 + b^2)}$

2.5.10 We have

$$\frac{\partial}{\partial x}(x + z)^2 + \frac{\partial}{\partial x}(y + z)^2 = \frac{\partial}{\partial x} 8 \implies 2(1 + \frac{\partial z}{\partial x})(x + z) + 2\frac{\partial z}{\partial x}(y + z) = 0.$$

At $(1, 1, 1)$ the last equation becomes

$$4(1 + \frac{\partial z}{\partial x}) + 4\frac{\partial z}{\partial x} = 0 \implies \frac{\partial z}{\partial x} = -\frac{1}{2}.$$

2.6.1 $\nabla f(x, y, z) = \begin{bmatrix} e^{yz} \\ xze^{yz} \\ xye^{yz} \end{bmatrix} \implies (\nabla f)(2, 1, 1) = \begin{bmatrix} e \\ 2e \\ 2e \end{bmatrix}.$

2.6.2 $(\nabla \times f)(x, y, z) = \begin{bmatrix} 0 \\ x \\ ye^{xy} \end{bmatrix} \implies (\nabla \times f)(2, 1, 1) = \begin{bmatrix} 0 \\ 2 \\ e^2 \end{bmatrix}.$

2.6.4 The vector $\begin{bmatrix} 1 \\ -7 \\ 0 \end{bmatrix}$ is perpendicular to the plane. Put $f(x, y, z) = x^2 + y^2 - 5xy + xz - yz + 3$. Then $(\nabla f)(x, y, z) = \begin{bmatrix} 2x - 5y + z \\ 2y - 5x - z \\ x - y \end{bmatrix}$. Observe that $\nabla f(x, y, z)$ is parallel to the vector $\begin{bmatrix} 1 \\ -7 \\ 0 \end{bmatrix}$, and hence there exists a constant a such that

$$\begin{bmatrix} 2x - 5y + z \\ 2y - 5x - z \\ x - y \end{bmatrix} = a \begin{bmatrix} 1 \\ -7 \\ 0 \end{bmatrix} \implies x = a, \quad y = a, \quad z = 4a.$$

Since the point is on the plane

$$x - 7y = -6 \implies a - 7a = -6 \implies a = 1.$$

Thus $x = y = 1$ and $z = 4$.

2.6.7 Observe that

$$f(0, 0) = 1, \quad f_x(x, y) = (\cos 2y)e^{x \cos 2y} \implies f_x(0, 0) = 1,$$
$$f_y(x, y) = -2x \sin 2ye^{x \cos 2y} \implies f_y(0, 0) = 0.$$

Hence

$$f(x, y) \approx f(0, 0) + f_x(0, 0)(x - 0) + f_y(0, 0)(y - 0) \implies f(x, y) \approx 1 + x.$$

This gives $f(0.1, -0.2) \approx 1 + 0.1 = 1.1$.

2.6.8 This is essentially the product rule: $duv = udv + vdu$, where ∇ acts the differential operator and $\times$ is the product. Recall that when we defined the volume of a parallelepiped spanned by the vectors $\overrightarrow{a}$, $\overrightarrow{b}$, $\overrightarrow{c}$, we saw that

$$\overrightarrow{a} \bullet (\overrightarrow{b} \times \overrightarrow{c}) = (\overrightarrow{a} \times \overrightarrow{b}) \bullet \overrightarrow{c}.$$

Treating $\nabla = \nabla_{\overrightarrow{u}} + \nabla_{\overrightarrow{v}}$ as a vector, first keeping $\overrightarrow{v}$ constant and then keeping $\overrightarrow{u}$ constant we then see that

$$\nabla_{\overrightarrow{u}} \bullet (\overrightarrow{u} \times \overrightarrow{v}) = (\overrightarrow{\nabla} \times \overrightarrow{u}) \bullet \overrightarrow{v}, \qquad \nabla_{\overrightarrow{v}} \bullet (\overrightarrow{u} \times \overrightarrow{v}) = -\nabla \bullet (\overrightarrow{v} \times \overrightarrow{u}) = -(\overrightarrow{\nabla} \times \overrightarrow{v}) \bullet \overrightarrow{u}.$$

Thus

$$\nabla \bullet (u \times v) = (\nabla_{\overrightarrow{u}} + \nabla_{\overrightarrow{v}}) \bullet (u \times v) = \nabla_{\overrightarrow{u}} \bullet (\overrightarrow{u} \times \overrightarrow{v}) + \nabla_{\overrightarrow{v}} \bullet (\overrightarrow{u} \times \overrightarrow{v}) = (\overrightarrow{\nabla} \times \overrightarrow{u}) \bullet \overrightarrow{v} - (\overrightarrow{\nabla} \times \overrightarrow{v}) \bullet \overrightarrow{u}.$$

2.6.11 An angle of $\frac{\pi}{6}$ with the x-axis and $\frac{\pi}{3}$ with the y-axis.

2.8.7 We have

$$(\nabla f)(x, y) = \begin{bmatrix} 4x^3 - 4(x - y) \\ 4y^3 + 4(x - y) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies 4x^3 = 4(x - y) = -4y^3 \implies x = -y.$$

Hence

$$4x^3 - 4(x - y) = 0 \implies 4x^3 - 8x = 0 \implies 4x(x^2 - 2) = 0 \implies x \in \{-\sqrt{2}, 0, \sqrt{2}\}.$$

Since $x = -y$, the critical points are thus $(-\sqrt{2}, \sqrt{2}), (0, 0), (\sqrt{2}, -\sqrt{2})$. The Hessian is now,

$$\mathcal{H}f(x, y) = \begin{bmatrix} 12x^2 - 4 & 4 \\ 4 & 12y^2 - 4 \end{bmatrix},$$

and its principal minors are $\Delta_1 = 12x^2 - 4$ and $\Delta_2 = (12x^2 - 4)(12y^2 - 4) - 16$.

If $(x, y) = (-\sqrt{2}, \sqrt{2})$ or $(x, y) = (\sqrt{2}, -\sqrt{2})$, then $\Delta_1 = 20 > 0$ and $\Delta_2 = 384 > 0$, so the matrix is positive definite and we have a local minimum at each of these points.

If $(x, y) = (0, 0)$ then $\Delta_1 = -4 < 0$ and $\Delta_2 = 0$, so the matrix is negative semidefinite and further testing is needed. What happens in a neighbourhood of $(0, 0)$? We have

$$f(x, x) = 2x^4 > 0, \qquad f(x, -x) = 2x^4 - 4x^2 = 2x^2(x^2 - 1).$$

If x is close enough to 0, $= 2x^2(x^2 - 1) < 0$, which means that the function both increases and decreases to 0 in a neighbourhood of $(0, 0)$, meaning that there is a saddle point there.

2.8.8 We have

$$\nabla f(x,y,z) = \begin{bmatrix} 8xz - 2y - 8x \\ -2x + 1 \\ 4x^2 - 2z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \implies x = 1/2; \quad y = -1; \quad z = 1/2.$$

The hessian is

$$\mathscr{H} = \begin{bmatrix} 8z - 8 & -2 & 8x \\ -2 & 0 & 0 \\ 8x & 0 & -2 \end{bmatrix}.$$

The principal minors are $8z - 8$; -4, and 8. At $z = 1/2$, the matrix is negative definite and the critical point is thus a saddle point.

2.8.9 We have

$$(\nabla f)(x,y,z) = \begin{bmatrix} 2x + yz \\ 2y + xz \\ 2z + xy \end{bmatrix},$$

and

$$\mathscr{H}_r f = \begin{bmatrix} 2 & z & y \\ z & 2 & x \\ y & x & 2 \end{bmatrix}.$$

We see that $\Delta_1(x,y,z) = 2, \Delta_2(x,y,z) = \det \begin{bmatrix} 2 & z \\ z & 2 \end{bmatrix} = 4 - z^2$ and $\Delta_3(x,y,z) = \det \mathscr{H}_r f = 8 - 2x^2 - 2y^2 - 2z^2 + 2xyz$.

If $(\nabla f)(x,y,z) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ then we must have

$$\begin{aligned} 2x &= -yz, \\ 2y &= -xz, \\ 2z &= -xy, \end{aligned}$$

and upon multiplication of the three equations,

$$8xyz = -x^2y^2z^2,$$

that is,

$$xyz(xyz + 8) = 0.$$

Clearly, if $xyz = 0$, then we must have at least one of the variables equalling 0, in which case, by virtue of the original three equations, all equal 0. Thus $(0,0,0)$ is a critical point. If $xyz = -8$, then none of the variables is 0, and solving for x, say, we must have $x = -\dfrac{8}{yz}$, and substituting this into $2x + yz = 0$ we gather $(yz)^2 = 16$, meaning that either $yz = 4$, in which case $x = -2$, or $zy = -4$, in which case $x = 2$. It is easy to see then that either exactly one of the variables is negative, or all three are negative. The other critical points are therefore $(-2,2,2)$, $(2,-2,2)$, $(2,2,-2)$, and $(-2,-2,-2)$.

At $(0,0,0)$, $\Delta_1(0,0,0) = 2 > 0, \Delta_2(0,0,0) = 4 > 0, \Delta_1(0,0,0) = 8 > 0$, and thus $(0,0,0)$ is a minimum point. If $x^2 = y^2 = z^2 = 4, xyz = -8$, then $\Delta_2(x,y,z) = 0, \Delta_3 = -32$, so these points are saddle points.

2.8.10 We have

$$(\nabla f)(x,y,z) = \begin{bmatrix} 2xy + 2 \\ x^2 + 2yz \\ y^2 - 1 \end{bmatrix},$$

and

$$\mathscr{H}_r f = \begin{bmatrix} 2y & 2x & 0 \\ 2x & 2z & 2y \\ 0 & 2y & 0 \end{bmatrix}.$$

We see that $\Delta_1(x,y,z) = 2y$, $\Delta_2(x,y,z) = \det \begin{bmatrix} 2y & 2x \\ 2x & 2z \end{bmatrix} = 4yz - 4x^2$ and $\Delta_3(x,y,z) = \det \mathscr{H}_r f = -8y^3$.

If $(\nabla f)(x,y,z) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ then we must have

$$\begin{aligned} xy &= -1, \\ x^2 &= -2yz, \\ y &= \pm 1, \end{aligned}$$

and hence $(1,-1,\frac{1}{2})$, and $(-1,1,-\frac{1}{2})$ are the critical points. Now, $\Delta_1(1,-1,\frac{1}{2}) = -2$, $\Delta_2(1,-1,\frac{1}{2}) = -6$, and $\Delta_3(1,-1,\frac{1}{2}) = 8$. Thus $(1,-1,\frac{1}{2})$ is a saddle point. Similarly, $\Delta_1(-1,1,-\frac{1}{2}) = 2$, $\Delta_2(-1,1,-\frac{1}{2}) = -6$, and $\Delta_3(-1,1,-\frac{1}{2}) = -8$, shewing that $(1,-1,\frac{1}{2})$ is also a saddle point.

2.8.11 We find

$$\nabla f(x,y,z) = \begin{bmatrix} 4yz - 4x^3 \\ 4xz - 4y^3 \\ 4xy - 4z^3 \end{bmatrix}.$$

Assume $\nabla f(x,y,z) = \mathbf{0}$. Then

$$4yz = 4x^3,\ 4xz = 4y^3,\ 4xy = 4z^3 \implies xyz = x^4 = y^4 = z^4.$$

Thus $xyz \geq 0$, and if one of the variables is 0 so are the other two. Thus $(0,0,0)$ is the only critical point with at least one of the variables 0. Assume now that $xyz \neq 0$. Then

$$(xyz)^3 = x^4y^4z^4 = (xyz)^4 \implies xyz = 1 \implies yz = \frac{1}{x} \implies x^4 = 1 \implies x = \pm 1.$$

Similarly, $y = \pm 1$, $z = \pm 1$. Since $xyz = 1$, exactly two of the variables can be negative. Thus we find the following critical points:

$$(0,0,0), (1,1,1), (-1,-1,1), (-1,1,-1), (1,-1,-1).$$

The Hessian is

$$\mathscr{H}_x f = \begin{bmatrix} -12x^2 & 4z & 4y \\ 4z & -12y^2 & 4x \\ 4y & 4x & -12z^2 \end{bmatrix}.$$

If $1 = xyz = x^2 = y^2 = z^2$, we have $\Delta_1 = -12x^2 = -12 < 0$, $\Delta_2 = 144x^2y^2 - 16z^2 = 144 - 16 = 128 > 0$, and

$$\begin{aligned} \Delta_3 &= -1728x^2y^2z^2 + 192x^4 + 192z^4 + 128zyx + 192y^4 \\ &= -1728 + 192 + 192 + 128 + 192 \\ &= -1024 \\ &< 0. \end{aligned}$$

This means that for $xyz \neq 0$ the Hessian is negative definite and the function has a local maximum at each of the four points $(1,1,1), (-1,-1,1), (-1,1,-1), (1,-1,-1)$. Observe that at these critical points $f = 1$. Now $f(0,0,0) = 0$ and $f(-1,1,1) = -7$.

2.8.12 Rewrite: $f(x,y,z) = xyz(4-x-y-z) = 4xyz - x^2yz - xy^2z - xyz^2$. Then,

$$(\nabla f)(x,y,z) = \begin{bmatrix} 4yz - 2xyz - y^2z - yz^2 \\ 4xz - x^2z - 2xyz - xz^2 \\ 4xy - x^2y - xy^2 - 2xyz \end{bmatrix},$$

$$\mathcal{H}f(x,y,z) = \begin{bmatrix} -2yz & z(4-2x-2y-z) & y(4-2x-y-2z) \\ z(4-2x-2y-z) & -2xz & x(4-x-2y-2z) \\ y(4-2x-y-2z) & x(4-x-2y-2z) & -2xy \end{bmatrix}$$

Equating the gradient to zero, we obtain,

$$yz(4-2x-y-z) = 0, \quad xz(4-x-2y-z) = 0, \quad xy(4-x-y-2z) = 0.$$

If $xyz \neq 0$ then we must have

$$4 - 2x - y - z = 0, \quad 4 - x - 2y - z = 0, \quad 4 - x - y - 2z = 0 \implies x = y = z = 1.$$

In this case

$$\mathcal{H}f(1,1,1) = \begin{bmatrix} -2 & -1 & -1 \\ -1 & -2 & -1 \\ -1 & -1 & -2 \end{bmatrix}$$

and the principal minors are $\Delta_1 = -2 < 0$, $\Delta_2 = 3 > 0$, and $\Delta_3 = -4 < 0$, so the matrix is negative definite and we have a local maximum at $(1, 1, 1)$.

If either of x, y, or z is 0, we will get $\Delta_3 = 0$, so further testing is needed. Now,

$$f(x, x, x) = x^3(4 - 3x), \qquad f(x, -x, x) = x^3(-4 + x).$$

Thus as $x \to 0+$ then $f(x, x, x) > 0$ and $f(x, -x, x) < 0$, which means that in some neighbourhood of $(0, 0, 0)$ the function is both decreasing towards 0 and increasing towards 0, which means that $(0, 0, 0)$ is a saddle point.

2.8.13 To facilitate differentiation observe that $g(x, y, z) = (xe^{-x^2})(ye^{-y^2})(ze^{-z^2})$. Now

$$\nabla g(x, y, z) = \begin{bmatrix} (1 - 2x^2)(yz)(e^{-x^2})(e^{-y^2})(e^{-z^2}) \\ (1 - 2y^2)(xz)(e^{-x^2})(e^{-y^2})(e^{-z^2}) \\ (1 - 2z^2)(xy)(e^{-x^2})(e^{-y^2})(e^{-z^2}) \end{bmatrix}.$$

The function is 0 if any of the variables is 0. Since the function clearly assumes positive and negative values, we can discard any point with a 0. If $\nabla(x, y, z) = 0$, then $x = \pm\frac{1}{\sqrt{2}}$; $y = \pm\frac{1}{\sqrt{2}}$ $z = \pm\frac{1}{\sqrt{2}}$. We find

$$\mathscr{H}_x g = t(x, y, z) \begin{bmatrix} (4x^3 - 6x)(yz) & (1 - 2x^2)(1 - 2y^2)z & (1 - 2x^2)(1 - 2z^2)y \\ (1 - 2y^2)(1 - 2x^2)z & (4y^3 - 6y)(xz) & (1 - 2y^2)(1 - 2z^2)x \\ (1 - 2z^2)(1 - 2x^2)y & (1 - 2z^2)(1 - 2y^2)x & (4z^3 - 6z)(xy) \end{bmatrix},$$

with $t(x, y, z) = (e^{-x^2})(e^{-y^2})(e^{-z^2})$. Since at the critical points we have $1 - 2x^2 = 1 - 2y^2 = 1 - 2z^2 = 0$, the Hessian reduces to

$$\mathscr{H}_x g = (e^{-3/2}) \begin{bmatrix} (4x^3 - 6x)(yz) & 0 & 0 \\ 0 & (4y^3 - 6y)(xz) & 0 \\ 0 & 0 & (4z^3 - 6z)(xy) \end{bmatrix}.$$

We have

$$\Delta_1 = (4x^3 - 6x)(yz)$$

$$\Delta_2 = (4x^3 - 6x)(4y^3 - 6y)(xyz^2)$$

$$\Delta_3 = (4x^3 - 6x)(4y^3 - 6y)(4z^3 - 6z)(x^2y^2z^2).$$

Also,

$$4\left(\frac{1}{\sqrt{2}}\right)^3 - 6\left(\frac{1}{\sqrt{2}}\right) = -2\sqrt{2} < 0, \quad 4\left(-\frac{1}{\sqrt{2}}\right)^3 - 6\left(-\frac{1}{\sqrt{2}}\right) = 2\sqrt{2} > 0.$$

This means that if an even number of the variables is negative (0 or 2), then we the Hessian is negative definite, and if an odd numbers of the variables is positive (1 or 3), the Hessian is positive definite. We conclude that we have local maxima at

$$(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$$

and local minima at

$$(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}), (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}), (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}).$$

2.8.14 By the Fundamental Theorem of Calculus, there exists a continuously differentiable function G such that

$$f(x,y) = \int_{y^2-x}^{x^2+y} g(t)\mathrm{d}t = G(x^2+y) - G(y^2-x).$$

Hence

$$\begin{aligned}\frac{\partial f}{\partial x}(x,y) &= 2xG'(x^2+y) + G'(y^2-x) = 2xg(x^2+y) + g(y^2-x);\\ \frac{\partial f}{\partial y}(x,y) &= G'(x^2+y) - 2yG'(y^2-x) = g(x^2+y) - 2yg(y^2-x).\end{aligned}$$

This gives

$$\frac{\partial f}{\partial x}(0,0) = g(0) = \frac{\partial f}{\partial y}(0,0) = 0,$$

so $(0,0)$ is a critical point. Now, the Hessian of f is

$$\mathscr{H}_f(x,y) = \begin{bmatrix} 2g(x^2+y) + 4x^2g'(x^2+y) - g'(y^2-x) & 2xg'(x^2+y) + 2yg'(y^2-x) \\ 2xg'(x^2-y) + 2yg'(y^2-x) & g'(x^2+y) - 2g(y^2-x) - 4y^2g'(y^2-x)\end{bmatrix},$$

and so

$$\mathscr{H}_f(0,0) = \begin{bmatrix} -g'(0) & 0 \\ 0 & g'(0)\end{bmatrix}.$$

Regardless of the sign of $g'(0)$, the determinant of of this last matrix is $-(g'(0))^2 < 0$, and so $(0,0)$ is a saddle point.

2.8.15 Since the coordinates $(x, \frac{\sqrt{144-16x^2}}{3})$, $-3 \leq x \leq 3$ describe an ellipse centred at the origin and semi-axes 3 and 4, and the coordinates $(y, \sqrt{4-y^2})$, $-2 \leq y \leq 2$ describe a circle centred at the origin with radius 2, the problem reduces to finding the minimum between the boundaries of the circle and the ellipse. Geometrically this is easily seen to be 1.

2.9.1 We have

$$\begin{aligned}\nabla(abc) = \lambda\nabla(2ab+2bc+2ca-S) \quad &\implies \quad \begin{bmatrix} bc \\ ca \\ ab\end{bmatrix} = \lambda\begin{bmatrix} 2b+2c \\ 2a+2c \\ 2b+2a\end{bmatrix}\\ &\implies \quad \begin{array}{rcl} bc & = & 2\lambda(b+c) \\ ca & = & 2\lambda(a+c) \\ ab & = & 2\lambda(b+a)\end{array}\end{aligned}$$

By physical considerations, $abc \neq 0$ and so $\lambda \neq 0$. Hence, by successively dividing the equations,

$$\frac{b}{a} = \frac{b+c}{c+a} \implies a = b, \quad \frac{c}{b} = \frac{a+c}{b+a} \implies b = c, \quad \frac{a}{c} = \frac{b+a}{b+c} \implies a = c.$$

Therefore

$$2a^2 + 2a^2 + 2a^2 = S \implies a = \frac{\sqrt{S}}{\sqrt{6}},$$

and the maximum volume is

$$abc = \frac{(\sqrt{S})^3}{(\sqrt{6})^3}.$$

The above result can be simply obtained by using the AM-GM inequality:

$$\frac{S}{3} = \frac{2ab+2bc+2ca}{3} \geq ((2ab)(2bc)(2ca))^{1/3} = 2(abc)^{2/3} \implies abc \leq \frac{S^{3/2}}{6^{3/2}}.$$

Equality happens if

$$2ab = 2bc = 2ca \implies a = b = c = \frac{\sqrt{S}}{\sqrt{6}}.$$

2.9.2

1. The vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is perpendicular to $\mathbf{\Pi}$. Hence, the equation of the perpendicular passing through P is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + t \begin{bmatrix} a \\ b \\ c \end{bmatrix} \implies x = 1 + ta, \quad y = 1 + tb, \quad z = 1 + tc.$$

The intersection of the line and the plane happens when

$$a(1 + at) + b(1 + tb) + c(1 + tc) = d \implies t = \frac{d - a - b - c}{a^2 + b^2 + c^2}.$$

Hence

$$P' = \begin{pmatrix} 1 + a \cdot \dfrac{d - a - b - c}{a^2 + b^2 + c^2} \\ 1 + b \cdot \dfrac{d - a - b - c}{a^2 + b^2 + c^2} \\ 1 + c \cdot \dfrac{d - a - b - c}{a^2 + b^2 + c^2} \end{pmatrix}$$

The distance is then

$$\sqrt{\left(a \cdot \frac{d - a - b - c}{a^2 + b^2 + c^2}\right)^2 + \left(b \cdot \frac{d - a - b - c}{a^2 + b^2 + c^2}\right)^2 + \left(c \cdot \frac{d - a - b - c}{a^2 + b^2 + c^2}\right)^2}$$

2. Let $f(x, y, z) = (x - 1)^2 + (y - 1)^2 + (z - 1)^2$ be the square of the distance from P to a point on the plane and let $g(x, y, z) = ax + by + cz - d$. Using Lagrange multipliers,

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \implies \begin{bmatrix} 2(x - 1) \\ 2(y - 1) \\ 2(z - 1) \end{bmatrix} = \lambda \begin{bmatrix} a \\ b \\ c \end{bmatrix} \implies 2(x-1) = \lambda a, \quad 2(y-1) = \lambda b, \quad 2(z-1) = \lambda c.$$

Since $(1, 1, 1)$ is not on the plane and $abc \neq 0$, we gather that $\lambda \neq 0$. Now,

$$x = 1 + \frac{\lambda a}{2}, \quad y = 1 + \frac{\lambda b}{2}, \quad z = 1 + \frac{\lambda c}{2}.$$

Putting these into the equation of the plane,

$$a\left(1 + \frac{\lambda a}{2}\right) + b\left(1 + \frac{\lambda b}{2}\right) + c\left(1 + \frac{\lambda c}{2}\right) = d \implies \lambda = 2 \cdot \frac{d - a - b - c}{a^2 + b^2 + c^2}.$$

Then the coordinates of P' are

$$x = 1 + \frac{\lambda a}{2} = 1 + a \cdot \frac{d - a - b - c}{a^2 + b^2 + c^2}, \quad y = 1 + \frac{\lambda b}{2} = 1 + b \cdot \frac{d - a - b - c}{a^2 + b^2 + c^2}, \quad z = 1 + \frac{\lambda c}{2} = 1 + c \cdot \frac{d - a - b - c}{a^2 + b^2 + c^2},$$

as before.

3. Consider the function

$$t(x, y) = (x - 1)^2 + (y - 1)^2 + \left(\frac{d - ax - by}{c} - 1\right)^2,$$

which is the square of the distance from a point (x, y, z) on the plane to the point $(1, 1, 1)$.
Now,

$$\nabla t(x, y) = \begin{bmatrix} 2(x - 1) - 2\dfrac{a}{c}\left(\dfrac{d - ax - by}{c} - 1\right) \\ 2(y - 1) - 2\dfrac{b}{c}\left(\dfrac{d - ax - by}{c} - 1\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

which implies

$$x = \frac{-b^2 - c^2 + ab - ad + ac}{a^2 + b^2 + c^2} = 1 + a \cdot \frac{d - a - b - c}{a^2 + b^2 + c^2}, \qquad y = \frac{c^2 + a^2 - ab + bd - bc}{a^2 + b^2 + c^2} = 1 + b \cdot \frac{d - a - b - c}{a^2 + b^2 + c^2},$$

as before. Substituting this in the equation of the plane gives the same coordinate of z, as before.

2.9.3 Using CBS,

$$\frac{x+3y}{2} \leq (\frac{x^4+81y^4}{2})^{1/4} = \frac{36^{1/4}}{2^{1/4}} \implies x+3y \leq 2^{3/4}\sqrt{6} = 2^{5/4}\sqrt{3}.$$

2.9.4 Using AM-GM,

$$\frac{1}{6^{1/3}} = \sqrt[6]{x^2y^3z} \leq \frac{2x+3y+z}{6} \implies 2x+3y+z \geq 6^{2/3}.$$

2.9.5 We put $g(x,y) = 5x^2+6xy+5y^2-8$ and argue using Lagrange multipliers. We have

$$\nabla f(x,y) = \lambda \nabla g(x,y) \implies \begin{bmatrix} 2x \\ 2y \end{bmatrix} = \lambda \begin{bmatrix} 10x+6y \\ 6x+10y \end{bmatrix}.$$

This gives the three equations

$$0 = 5(\lambda-1)x+3y; \quad 0 = 3x+5(\lambda-1)y; \quad 5x^2+6xy+5y^2 = 8.$$

The linear system (the first two equations) will have the unique solution $(0,0)$ as long as $25(\lambda-1)^2-9 \neq 0$, but this solution does not lie on the third equation. If $25(\lambda-1)^2-9=0$, then we deduce that $x = \pm y$. Substituting this into the third equation we gather that $10x^2 \pm 6x^2 = 8$, resulting in $x = \pm\sqrt{2}$ or $x = \pm\frac{1}{\sqrt{2}}$. Taking into account the third equation, the feasible values are $(\sqrt{2}, -\sqrt{2})$, $(-\sqrt{2}, \sqrt{2})$, $(1/\sqrt{2}, 1/\sqrt{2})$, $(-1/\sqrt{2}, -1/\sqrt{2})$ The desired maximum is thus

$$f(-\sqrt{2}, \sqrt{2}) = f(\sqrt{2}, -\sqrt{2}) = 4$$

and the minimum is

$$f(1/\sqrt{2}, 1/\sqrt{2}) = f(-1/\sqrt{2}, -1/\sqrt{2}) = 1.$$

Aliter: Observe that, using AM-GM,

$$5x^2+6xy+5y^2 = 8 \implies x^2+y^2 = \frac{8}{5} - \frac{6}{5}xy \geq \frac{8}{5} - \frac{6}{5}\cdot\frac{x^2+y^2}{2} \implies x^2+y^2 \geq \frac{5}{8}\cdot\frac{8}{5} = 1.$$

2.9.6 Put $g(x,y) = x^p+y^p-1$. We need $a = p\lambda x^{p-1}$ and $b = p\lambda y^{p-1}$. Clearly then , $\lambda \neq 0$. We then have

$$x = \left(\frac{a}{\lambda p}\right)^{1/(p-1)}, \quad y = \left(\frac{b}{\lambda p}\right)^{1/(p-1)}.$$

Thus

$$1 = x^p + y^p = \left(\frac{a}{\lambda p}\right)^{p/(p-1)} + \left(\frac{b}{\lambda p}\right)^{p/(p-1)},$$

which gives

$$\lambda = \left(\left(\frac{a}{p}\right)^{p/(p-1)} + \left(\frac{b}{p}\right)^{p/(p-1)}\right)^{(p-1)/p}.$$

This gives

$$x = \frac{a^{1/(p-1)}}{(a^{1/(p-1)}+b^{1/(p-1)})^{1/p}}, \quad y = \frac{b^{1/(p-1)}}{(a^{1/(p-1)}+b^{1/(p-1)})^{1/p}}.$$

Since f is non-negative, these points define a maximum for f and so

$$ax+by \leq \frac{a^{p/(p-1)}}{(a^{1/(p-1)}+b^{1/(p-1)})^{1/p}} + \frac{b^{p/(p-1)}}{(a^{1/(p-1)}+b^{1/(p-1)})^{1/p}}.$$

2.9.7 Let $g(x,y,z) = (x-1)^2+(y-2)^2+(z-3)^2-4$. We solve

$$\nabla f \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \lambda \nabla g \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

for x, y, λ. This requires

$$\begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} = \begin{bmatrix} 2(x-1)\lambda \\ 2(y-2)\lambda \\ 2(z-3)\lambda \end{bmatrix}.$$

Clearly, $\lambda \neq 1$. This gives $x = \dfrac{-\lambda}{1-\lambda}$, $y = \dfrac{-2\lambda}{1-\lambda}$, and $z = \dfrac{-3\lambda}{1-\lambda}$. Substituting into $(x-1)^2+(y-2)^2+(z-3)^2 = 4$, we gather that

$$\left(\frac{-\lambda}{1-\lambda}-1\right)^2+\left(\frac{-2\lambda}{1-\lambda}-2\right)^2+\left(\frac{-3\lambda}{1-\lambda}-3\right)^2 = 4,$$

from where

$$\lambda = 1 \pm \frac{\sqrt{14}}{2}.$$

This gives the two points

$$(x, y, z) = \left(1+\frac{2}{\sqrt{14}}, 2+\frac{4}{\sqrt{14}}, 3+\frac{6}{\sqrt{14}}\right)$$

and

$$(x, y, z) = \left(1-\frac{2}{\sqrt{14}}, 2-\frac{4}{\sqrt{14}}, 3-\frac{6}{\sqrt{14}}\right).$$

The first point gives an absolute maximum of $18+\dfrac{12\sqrt{14}}{7}$ and the second an absolute minimum of $18-\dfrac{12\sqrt{14}}{7}$.

2.9.8 Observe that the ellipse is symmetric about the origin. Now maximise and minimise the distance between a point on the ellipse and the origin. If a and b are the semi-axes, you will find that $2a = 2$ and $2b = 6$

2.9.9 Put $g(x, y, z) = x^2 + y^2 - 2, h(x, y, z) = x + z - 1$. We must find λ, δ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) + \delta \nabla h(x, y, z),$$

which translates into

$$\begin{aligned} 1 &= 2\lambda x + \delta, \\ 1 &= 2\lambda y, \\ 1 &= \delta, \end{aligned}$$

and

$$\begin{aligned} x^2 + y^2 &= 1, \\ x + z &= 1. \end{aligned}$$

We deduce that $x = 0, y = \pm\sqrt{2}, z = 1$. We may shew that $(0, \sqrt{2}, 1)$ yields a maximum and that $(0, -\sqrt{2}, 1)$ yields a minimum.

2.9.10 One can use Lagrange multipliers here. But perhaps the easiest approach is to put $y = 1-x$ and maximise

$$f(x) = x + \sqrt{x(1-x)}.$$

For this we have

$$f'(x) = 0 \implies 1 + \frac{1-2x}{2\sqrt{x(1-x)}} = 0 \implies x = \frac{1}{2} + \frac{\sqrt{2}}{4}.$$

Since

$$f''(x) = -\frac{(1-2x)^2}{4(x(1-x))^{3/2}} - \frac{1}{\sqrt{x(1-x)}} < 0,$$

the value sought is a maximum. This maximum is thus

$$f\left(\frac{1}{2}+\frac{\sqrt{2}}{4}\right) = \frac{1}{2}+\frac{\sqrt{2}}{2}.$$

2.9.11 Claim: the function achieves its maximum on the boundary of the triangle. To prove this claim we have to prove that there are no critical points strictly inside the triangle. For this we compute the gradient and set it equal to the zero vector:

$$(\nabla f)(x, y) = \begin{bmatrix} -ax^{a-1}y^b e^{-(x+y)} \\ -bx^a y^{b-1} e^{-(x+y)} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies x = 0 \text{ or } y = 0,$$

which means that the critical points occur on the boundary. Since the function is identically 0 for $x = 0$ or $y = 0$, we only need to look on the line $x + y = 1$ for the maxima. Hence we maximise f subject to the constraint $x + y = 1$. Since $x + y = 1$, we can see that $f(x, y) = x^a y^b e^{-(x+y)} = x^a y^b e^{-1}$ on the line, so the problem reduces to maximising $h(x, y) = x^a y^b$ subject to the constraint $x + y = 1$. Using Lagrange multipliers,

$$(\nabla h)(x, y) = \lambda(\nabla g)(x, y) \implies \begin{bmatrix} ax^{a-1}y^b \\ bx^a y^{b-1} \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

which in turn

$$\implies ax^{a-1}y^b = \lambda = bx^a y^{b-1} \implies ay = bx \implies ay = b(1-y) \implies y = \frac{b}{a+b}, \quad x = \frac{a}{a+b}.$$

Finally,

$$f(x,y) = x^a y^b e^{-(x+y)} \le x^a y^b e^{-1} \le \left(\frac{a}{a+b}\right)^a \left(\frac{b}{a+b}\right)^b e^{-1}.$$

2.9.15 Try $p(x,y) = (y^2+1)x^2 + 2xy + 1$.

3.3.1

1. Let $L_1 : y = x+1$, $L_2 : -x+1$. Then

$$\begin{aligned}\int_C x\mathrm{d}x + y\mathrm{d}y &= \int_{L_1} x\mathrm{d}x + y\mathrm{d}y + \int_{L_2} x\mathrm{d}x + y\mathrm{d}y \\ &= \int_{-1}^{1} x\mathrm{d}x(x+1)\mathrm{d}x + \int_0^1 x\mathrm{d}x - (-x+1)\mathrm{d}x \\ &= 0.\end{aligned}$$

Also, both on L_1 and on L_2 we have $||\mathrm{d}\mathbf{x}|| = \sqrt{2}\mathrm{d}x$, thus

$$\begin{aligned}\int_C xy||\mathrm{d}\mathbf{x}|| &= \int_{L_1} xy||\mathrm{d}\mathbf{x}|| + \int_{L_2} xy||\mathrm{d}\mathbf{x}|| \\ &= \sqrt{2}\int_{-1}^{1} x(x+1)\mathrm{d}x - \sqrt{2}\int_0^1 x(-x+1)\mathrm{d}x \\ &= 0.\end{aligned}$$

2. We put $x = \sin t$, $y = \cos t$, $t \in \left[-\frac{\pi}{2}; \frac{\pi}{2}\right]$. Then

$$\begin{aligned}\int_C x\mathrm{d}x + y\mathrm{d}y &= \int_{-\pi/2}^{\pi/2} (\sin t)(\cos t)\mathrm{d}t - (\cos t)(\sin t)\mathrm{d}t \\ &= 0.\end{aligned}$$

Also, $||\mathrm{d}\mathbf{x}|| = \sqrt{(\cos t)^2 + (-\sin t)^2}\mathrm{d}t = \mathrm{d}t$, and thus

$$\begin{aligned}\int_C xy||\mathrm{d}\mathbf{x}|| &= \int_{-\pi/2}^{\pi/2} (\sin t)(\cos t)\mathrm{d}t \\ &= \left.\frac{(\sin t)^2}{2}\right|_{-\pi/2}^{\pi/2} \\ &= 0.\end{aligned}$$

3.3.2 Let Γ_1 denote the straight line segment path from O to $A = (2\sqrt{3}, 2)$ and Γ_2 denote the arc of the circle centred at $(0,0)$ and radius 4 going counterclockwise from $\theta = \frac{\pi}{6}$ to $\theta = \frac{\pi}{5}$.

Observe that the Cartesian equation of the line $\overleftrightarrow{\mathrm{OA}}$ is $y = \frac{x}{\sqrt{3}}$. Then on Γ_1

$$x\mathrm{d}x + y\mathrm{d}y = x\mathrm{d}x + \frac{x}{\sqrt{3}}\mathrm{d}\frac{x}{\sqrt{3}} = \frac{4}{3}x\mathrm{d}x.$$

Hence

$$\int_{\Gamma_1} x\mathrm{d}x + y\mathrm{d}y = \int_0^{2\sqrt{3}} \frac{4}{3}x\mathrm{d}x = 8.$$

On the arc of the circle we may put $x = 4\cos\theta$, $y = 4\sin\theta$ and integrate from $\theta = \frac{\pi}{6}$ to $\theta = \frac{\pi}{5}$. Observe that there

$$x\mathrm{d}x + y\mathrm{d}y = (\cos\theta)\mathrm{d}\cos\theta + (\sin\theta)\mathrm{d}\sin\theta = -\sin\theta\cos\theta\mathrm{d}\theta + \sin\theta\cos\theta\mathrm{d}\theta = 0,$$

and since the integrand is 0, the integral will be zero.

Assembling these two pieces,

$$\int_{\Gamma} x\mathrm{d}x + y\mathrm{d}y = \int_{\Gamma_1} x\mathrm{d}x + y\mathrm{d}y + \int_{\Gamma_2} x\mathrm{d}x + y\mathrm{d}y = 8 + 0 = 8.$$

To solve this problem using Maple you may use the code below.

```
> with(Student[VectorCalculus]):
> LineInt( VectorField( <x,y> ), Line( <0,0>, <2*sqrt(3),2> ))
> +LineInt( VectorField( <x,y> ), Arc(Circle( <0,0>, 4), Pi/6, Pi/5) );
```

3.3.3 Using the parametrisations from the solution of problem 3.3.3, we find on Γ_1 that

$$x||\mathrm{d}\mathbf{x}|| = x\sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = x\sqrt{1 + \frac{1}{3}}\mathrm{d}x = \frac{2}{\sqrt{3}}x\mathrm{d}x,$$

whence

$$\int_{\Gamma_1} x||\mathrm{d}\mathbf{x}|| = \int_0^{2\sqrt{3}} \frac{2}{\sqrt{3}}x\mathrm{d}x = 4\sqrt{3}.$$

On Γ_2 that

$$x||\mathrm{d}\mathbf{x}|| = x\sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2} = 16\cos\theta\sqrt{\sin^2\theta + \cos^2\theta}\mathrm{d}\theta = 16\cos\theta\mathrm{d}\theta,$$

whence

$$\int_{\Gamma_2} x||\mathrm{d}\mathbf{x}|| = \int_{\pi/6}^{\pi/5} 16\cos\theta\mathrm{d}\theta = 16\sin\frac{\pi}{5} - 16\sin\frac{\pi}{6} = 4\sin\frac{\pi}{5} - 8.$$

Assembling these we gather that

$$\int_{\Gamma} x||\mathrm{d}\mathbf{x}|| = \int_{\Gamma_1} x||\mathrm{d}\mathbf{x}|| + \int_{\Gamma_2} x||\mathrm{d}\mathbf{x}|| = 4\sqrt{3} - 8 + 16\sin\frac{\pi}{5}.$$

To solve this problem using Maple you may use the code below.

```
> with(Student[VectorCalculus]):
> PathInt(x, [x,y]= Line( <0,0>, <2*sqrt(3),2> ))
> +PathInt(x, [x,y]=Arc(Circle( <0,0>, 4), Pi/6, Pi/5) );
```

Maple gives $16\cos\frac{3\pi}{10}$ rather than our $16\sin\frac{\pi}{5}$. To check that these two are indeed the same, use the code

```
> is(16*cos(3*Pi/10)=16*sin(Pi/5));
```

which returns `true`.

3.3.4 The curve lies on the sphere, and to parametrise this curve, we dispose of one of the variables, y say, from where $y = 1 - x$ and $x^2 + y^2 + z^2 = 1$ give

$$\begin{aligned} x^2 + (1-x)^2 + z^2 = 1 \quad &\implies \quad 2x^2 - 2x + z^2 = 0 \\ &\implies \quad 2\left(x - \frac{1}{2}\right)^2 + z^2 = \frac{1}{2} \\ &\implies \quad 4\left(x - \frac{1}{2}\right)^2 + 2z^2 = 1. \end{aligned}$$

So we now put

$$x = \frac{1}{2} + \frac{\cos t}{2}, \quad z = \frac{\sin t}{\sqrt{2}}, \quad y = 1 - x = \frac{1}{2} - \frac{\cos t}{2}.$$

We must integrate on the side of the plane that can be viewed from the point $(1, 1, 0)$ (observe that the vector $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ is normal to the plane). On the zx-plane, $4\left(x - \frac{1}{2}\right)^2 + 2z^2 = 1$ is an ellipse. To obtain a positive parametrisation we must integrate from $t = 2\pi$ to $t = 0$ (this is because when you look at the ellipse from the point $(1, 1, 0)$ the

positive x-axis is to your left, and not your right). Thus

$$\begin{aligned}
\oint_{\Gamma} z\mathrm{d}x + x\mathrm{d}y + y\mathrm{d}z &= \int_{2\pi}^{0} \frac{\sin t}{\sqrt{2}} \mathrm{d}\left(\frac{1}{2} + \frac{\cos t}{2}\right) \\
&\quad + \int_{2\pi}^{0} \left(\frac{1}{2} + \frac{\cos t}{2}\right) \mathrm{d}\left(\frac{1}{2} - \frac{\cos t}{2}\right) \\
&\quad + \int_{2\pi}^{0} \left(\frac{1}{2} - \frac{\cos t}{2}\right) \mathrm{d}\left(\frac{\sin t}{\sqrt{2}}\right) \\
&= \int_{2\pi}^{0} \left(\frac{\sin t}{4} + \frac{\cos t}{2\sqrt{2}} + \frac{\cos t \sin t}{4} - \frac{1}{2\sqrt{2}}\right) \mathrm{d}t \\
&= \frac{\pi}{\sqrt{2}}.
\end{aligned}$$

3.5.1 2

3.5.2 $\frac{1}{3}$

3.5.3 $\frac{15\pi}{16}$

3.5.4 The integral equals

$$\begin{aligned}
\int_{D} xy\mathrm{d}x\mathrm{d}y &= \int_{0}^{1} x\left(\int_{x^2}^{\sqrt{x}} y \ \mathrm{d}y\right) \mathrm{d}x \\
&= \int_{0}^{1} \frac{1}{2} x(x - x^4) \ \mathrm{d}x \\
&= \frac{1}{12}.
\end{aligned}$$

3.5.5 The integral equals

$$\begin{aligned}
\int_{D} x\sin x \sin y\mathrm{d}x\mathrm{d}y + \int_{D} y \sin x \sin y \mathrm{d}x\mathrm{d}y &= 2\left(\int_{0}^{\pi} y\sin y \ \mathrm{d}y\right)\left(\int_{0}^{\pi} \sin x \ \mathrm{d}x\right) \\
&= 4\pi.
\end{aligned}$$

3.5.6 The integral is

$$\begin{aligned}
\int_{x\leq y} x^2\mathrm{d}x\mathrm{d}y + \int_{y\leq x} y^2\mathrm{d}x\mathrm{d}y &= \int_{0}^{1}\int_{0}^{y} x^2\mathrm{d}x\mathrm{d}y + \int_{0}^{1}\int_{y}^{1} y^2\mathrm{d}x\mathrm{d}y \\
&= \int_{0}^{1} \frac{y^3}{3}\mathrm{d}y + \int_{0}^{1} (y^2 - y^3)\,\mathrm{d}y \\
&= \frac{y^4}{12}\bigg|_{0}^{1} + \left(\frac{y^3}{3} - \frac{y^4}{4}\right)\bigg|_{0}^{1} \\
&= \frac{1}{12} + \frac{1}{3} - \frac{1}{4} \\
&= \frac{1}{6}.
\end{aligned}$$

3.5.7 $\frac{21}{8}$

3.5.8 Observe that

$$x^2 + y^2 = 16, y = -\frac{\sqrt{3}}{3}x + 4 \implies 16 - x^2 = \left(-\frac{\sqrt{3}}{3}x + 4\right)^2 \implies x = 0, 2\sqrt{3}.$$

The integral is

$$\begin{aligned}
\int_{0}^{2\sqrt{3}}\int_{-\frac{\sqrt{3}}{3}x+4}^{\sqrt{16-x^2}} x\mathrm{d}y\mathrm{d}x &= \int_{0}^{2\sqrt{3}} x\left(\sqrt{16 - x^2} + \frac{\sqrt{3}}{3}x - 4\right) \mathrm{d}x \\
&= -\frac{1}{3}(16 - x^2)^{3/2} + \frac{\sqrt{3}}{9}x^3 - 2x^2\bigg|_{0}^{2\sqrt{3}} \\
&= \frac{8}{3}.
\end{aligned}$$

3.5.9 $e-1$

3.5.10 We have

$$\begin{aligned}
\int\limits_{[0;1]^2} \min(x,y^2)\mathrm{d}A &= \int\limits_{\substack{[0;1]^2\\ x\le y^2}} x\mathrm{d}A + \int\limits_{\substack{[0;1]^2\\ y^2<x}} y^2\mathrm{d}A \\
&= \int_0^1\int_0^{y^2} x\mathrm{d}x\mathrm{d}y + \int_0^1\int_{y^2}^1 y^2\mathrm{d}x\mathrm{d}y \\
&= \frac{1}{2}\int_0^1 x^2\Big|_0^{y^2}\mathrm{d}y + \int_0^1 y^2x\Big|_{y^2}^1\mathrm{d}y \\
&= \frac{1}{2}\int_0^1 y^4\mathrm{d}y + \int_0^1 (y^2-y^4)\mathrm{d}y \\
&= \frac{1}{10}+\frac{2}{15} \\
&= \frac{7}{30}.
\end{aligned}$$

3.5.11 Begin by finding the Cartesian equations of the various lines: for $\overleftrightarrow{\mathrm{OA}}$ is $y=\dfrac{x}{3}$ $(0\le x\le 1)$, for $\overleftrightarrow{\mathrm{AB}}$ is $y=3x-8$ $(3\le x\le 4)$, and for $\overleftrightarrow{\mathrm{BO}}$ is $y=x$ $(0\le x\le 4)$.

We have a choice of whether integrating with respect to x or y first. Upon examining the region, one notices that it does not make much of a difference. I will integrate with respect to y first. In such a case notice that for $0\le x\le 3$, y goes from the line $\overleftrightarrow{\mathrm{OA}}$ to the line $\overleftrightarrow{\mathrm{OB}}$, and for $3\le x\le 4$, y goes from the line $\overleftrightarrow{\mathrm{AB}}$ to the line $\overleftrightarrow{\mathrm{OB}}$

$$\begin{aligned}
\int_{\mathcal{R}} xy\mathrm{d}A &= \int_0^3\int_{x/3}^x xy\mathrm{d}y\mathrm{d}x + \int_3^4\int_{3x-8}^x xy\mathrm{d}y\mathrm{d}x \\
&= \frac{1}{2}\int_0^3 xy^2\Big|_{x/3}^x\mathrm{d}x + \frac{1}{2}\int_3^4 xy^2\Big|_{3x-8}^x\mathrm{d}x \\
&= \frac{1}{2}\int_0^3 x\left(x^2-\frac{x^2}{9}\right)\mathrm{d}x + \frac{1}{2}\int_3^4 x\left(x^2-(3x-8)^2\right)\mathrm{d}x \\
&= \frac{4}{9}\int_0^3 x^3\mathrm{d}x + \frac{1}{2}\int_3^4\left(-8x^3+48x^2-64x\right)\mathrm{d}x \\
&= 9+9 \\
&= 18.
\end{aligned}$$

To solve this problem using Maple you may use the code below.

```
>  with(Student[VectorCalculus]):
>  int(x*y, [x,y]=Triangle(<0,0>,<3,1>,<4,4>));
```

Maple can also provide the limits of integration, but this command is limited, since Maple is quite whimsical about which order of integration to choose. It also evaluates expressions that it deems below its dignity to return unevaluated.

```
>  int(x*y, [x,y]=Triangle(<0,0>,<3,1>,<4,4>), 'inert');
```

3.5.12 Integrating by parts,

$$\begin{aligned}
\int\limits_D \log_e(1+x+y)\mathrm{d}x\mathrm{d}y &= \int_0^1\left(\int_0^{1-x}\log_e(1+x+y)\ \mathrm{d}y\right)\ \mathrm{d}x \\
&= \int_0^1 \left[(1+x+y)\log_e(1+x+y)-(1+x+y)\right]_0^{1-x}\ \mathrm{d}x \\
&= \int_0^1 \left(2\log_e(2)-1-\log_e(1+x)-x\log_e(1+x)+x\right)\ \mathrm{d}x \\
&= \frac{1}{4}.
\end{aligned}$$

3.5.13 First observe that on $[0;2]^2$, $0 \le \llbracket x+y^2 \rrbracket \le 6$, so we decompose the region of integration according to where $\llbracket x+y^2 \rrbracket$ jumps across integer values. We have

$$\begin{aligned}\int_{[0;2]^2} \llbracket x+y^2 \rrbracket \mathrm{d}A &= \int_{\substack{[0;2]^2 \\ \llbracket x+y^2 \rrbracket = 1}} 1\mathrm{d}A + \int_{\substack{[0;2]^2 \\ \llbracket x+y^2 \rrbracket = 2}} 2\mathrm{d}A + \int_{\substack{[0;2]^2 \\ \llbracket x+y^2 \rrbracket = 3}} 3\mathrm{d}A + \int_{\substack{[0;2]^2 \\ \llbracket x+y^2 \rrbracket = 4}} 4\mathrm{d}A + \int_{\substack{[0;2]^2 \\ \llbracket x+y^2 \rrbracket = 5}} 5\mathrm{d}A \\ &= \int_{\substack{[0;2]^2 \\ 1 \le x+y^2 < 2}} \mathrm{d}A + 2\int_{\substack{[0;2]^2 \\ 2 \le x+y^2 < 3}} \mathrm{d}A + 3\int_{\substack{[0;2]^2 \\ 3 \le x+y^2 < 4}} \mathrm{d}A + 4\int_{\substack{[0;2]^2 \\ 4 \le x+y^2 < 5}} \mathrm{d}A + 5\int_{\substack{[0;2]^2 \\ 5 \le x+y^2 < 6}} \mathrm{d}A\end{aligned}$$

By looking at the regions (as in figures A.13 through A.17 below) (I am omitting the details of the integrations, relying on Maple for the evaluations), we obtain

$$\int_{\substack{[0;2]^2 \\ 1 \le x+y^2 < 2}} \mathrm{d}A = \int_0^1 \int_{\sqrt{1-x}}^{\sqrt{2-x}} \mathrm{d}y\mathrm{d}x + \int_1^2 \int_0^{\sqrt{2-x}} \mathrm{d}y\mathrm{d}x = -\frac{4}{3} + \frac{4}{3}\sqrt{2} + \frac{2}{3} = -\frac{2}{3} + \frac{4}{3}\sqrt{2}.$$

$$2\int_{\substack{[0;2]^2 \\ 2 \le x+y^2 < 3}} \mathrm{d}A = 2\int_0^2 \int_{\sqrt{2-x}}^{\sqrt{3-x}} \mathrm{d}y\ \mathrm{d}x = 4\sqrt{3} - \frac{8}{3}\sqrt{2} - \frac{4}{3}.$$

$$3\int_{\substack{[0;2]^2 \\ 3 \le x+y^2 < 4}} \mathrm{d}A = 3\int_0^2 \int_{\sqrt{3-x}}^{\sqrt{4-x}} \mathrm{d}y\ \mathrm{d}x = 18 - 6\sqrt{3} - 4\sqrt{2}.$$

$$4\int_{\substack{[0;2]^2 \\ 4 \le x+y^2 < 5}} \mathrm{d}A = 4\int_0^1 \int_{\sqrt{4-x}}^{2} \mathrm{d}y\ \mathrm{d}x + 4\int_1^2 \int_{\sqrt{4-x}}^{\sqrt{5-x}} \mathrm{d}y\ \mathrm{d}x = -\frac{40}{3} + 8\sqrt{3} + \frac{64}{3} - 16\sqrt{3} + \frac{16}{3}\sqrt{2}.$$

$$5\int_{\substack{[0;2]^2 \\ 4 \le x+y^2 < 5}} \mathrm{d}A = 5\int_1^2 \int_{\sqrt{5-x}}^{5} \mathrm{d}y\ \mathrm{d}x = -\frac{50}{3} + 10\sqrt{3}.$$

Adding all the above, we obtain

$$\int_{[0;2]^2} \llbracket x+y^2 \rrbracket \mathrm{d}A = \frac{22}{3} + \frac{4}{3}\sqrt{3} - \frac{4}{3}\sqrt{2} \approx 7.7571.$$

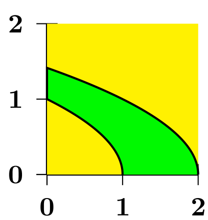

Figure A.13: $1 \le x + y^2 < 2$.

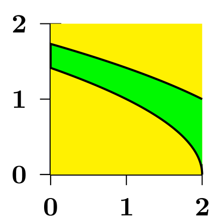

Figure A.14: $2 \le x + y^2 < 3$.

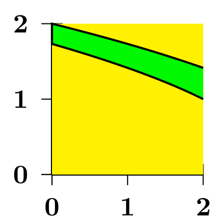

Figure A.15: $3 \le x + y^2 < 4$.

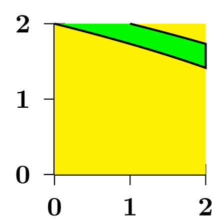

Figure A.16: $4 \le x + y^2 < 5$.

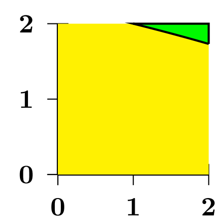

Figure A.17: $5 \le x + y^2 < 6$.

3.5.14 Observe that in the rectangle $[0\,;1] \times [0\,;2]$ we have $0 \le x+y \le 3$. Hence

$$\begin{aligned}\int_R \llbracket x+y \rrbracket \mathrm{d}A &= \int_{\substack{R \\ 1 \le x+y < 2}} 1\mathrm{d}A + \int_{\substack{R \\ 2 \le x+y < 3}} 2\mathrm{d}A \\ &= \int_1^2 \int_{1-x}^{2-x} 1\mathrm{d}y\mathrm{d}x + \int_1^2 \int_{2-x}^{2} 2\mathrm{d}y\mathrm{d}x \\ &= 4.\end{aligned}$$

3.5.15 $\displaystyle\int_0^1\int_{\sqrt{1-x^2}}^{\sqrt{4-x^2}} x\mathrm{d}y\mathrm{d}x + \int_1^2\int_0^{\sqrt{4-x^2}} x\mathrm{d}y\mathrm{d}x = \frac{7}{3}.$

3.5.16 $\displaystyle\int_0^1\int_{-2}^{3} x\mathrm{d}x\mathrm{d}y + \int_1^2\int_{-2}^{-1} x\mathrm{d}x\mathrm{d}y + \int_1^2\int_0^1 x\mathrm{d}x\mathrm{d}y + \int_1^2\int_2^3 x\mathrm{d}x\mathrm{d}y = 7.$

3.5.17 Exchanging the order of integration,

$$\int_0^{\pi/2}\int_0^y \frac{\cos y}{y}\mathrm{d}x\mathrm{d}y = \int_0^{\pi/2}\cos y\mathrm{d}y = 1.$$

3.5.18 Upon splitting the domain of integration, we find that the integral equals

$$\begin{aligned}\int_1^2\left(\int_y^{y^2}\sin\frac{\pi x}{2y}\mathrm{d}x\right)\mathrm{d}y &= \int_1^2\left[-\frac{2y}{\pi}\cos\frac{\pi x}{2y}\right]_y^{y^2}\mathrm{d}y\\ &= -\int_1^2 -\frac{2y}{\pi}\cos\frac{\pi y}{2}\mathrm{d}y\\ &= \frac{4(\pi+2)}{\pi^3},\end{aligned}$$

upon integrating by parts.

3.5.19 The integral is 0. Observe that if $(x,y)\in D$ then $(-x,y)\in D$. Also, $f(-x,y)=-f(x,y)$.

3.5.20 $\displaystyle\int_{-2/\sqrt{5}}^{2/\sqrt{5}}\int_{-\frac{1}{2}\sqrt{4-y^2}}^{\frac{1}{2}\sqrt{4-y^2}}\mathrm{d}x\mathrm{d}y = \frac{8}{5}+4\arcsin\left(\frac{\sqrt{5}}{5}\right).$

3.5.21 The integral equals

$$\begin{aligned}\int_D xy\mathrm{d}A &= \int_0^1\left(\int_0^{\frac{1-x}{1+x}} xy\ \mathrm{d}y\right)\mathrm{d}x\\ &= \int_0^1\left(\frac{1}{2}x\left(\frac{1-x}{1+x}\right)^2\right)\mathrm{d}x\\ &= \int_1^2\frac{(t-1)(t-2)^2}{t^2}\mathrm{d}t\\ &= 4\log_e 2-\frac{11}{4}.\end{aligned}$$

3.5.22 Using integration by parts,

$$\begin{aligned}\int_D \log_e(1+x^2+y)\mathrm{d}A &= \int_0^1\left(\int_0^{1-x^2}\log_e(1+x^2+y)\ \mathrm{d}y\right)\mathrm{d}x\\ &= \int_0^1(2\log_e(2)-1-\log_e(1+x^2))\mathrm{d}x\\ &\quad+\int_0^1(-x^2\log_e(1+x^2)+x^2)\ \mathrm{d}x\\ &= \frac{2}{3}\log_e 2+\frac{8}{9}-\frac{\pi}{3}.\end{aligned}$$

3.5.23 $\displaystyle\int_0^2\int_{\sqrt{2y-y^2}}^{\sqrt{4-y^2}} x\mathrm{d}x\mathrm{d}y = 2.$

3.5.25 Let

$$\begin{aligned}D_1 &= \{(x,y)\in\mathbb{R}^2 \mid -1\le x\le 1, x\le y\},\\ D_2 &= \{(x,y)\in\mathbb{R}^2 \mid -1\le x\le 1, x> y\}.\end{aligned}$$

Then $D=D_1\cup D_2, D_1\cap D_2=\emptyset$ and so

$$\int_D f(x,y)\mathrm{d}x\mathrm{d}y = \int_{D_1} f(x,y)\mathrm{d}x\mathrm{d}y + \int_{D_2} f(x,y)\mathrm{d}x\mathrm{d}y.$$

By symmetry,

$$\int_{D_1} f(x,y)\mathrm{d}x\mathrm{d}y = \int_{D_2} f(x,y)\mathrm{d}x\mathrm{d}y,$$

and so

$$\begin{aligned}\int_D f(x,y)\mathrm{d}x\mathrm{d}y &= 2\int_{D_1} f(x,y)\mathrm{d}x\mathrm{d}y \\ &= 2\int_{-1}^{1}\left(\int_x^1 (y-x)\ \mathrm{d}y\right)\ \mathrm{d}x \\ &= \int_{-1}^{1}(1-2x+x^2)\ \mathrm{d}x \\ &= \frac{8}{3}.\end{aligned}$$

3.5.26 The line joining A, and B has equation $y=-x-2$, line joining B, and C has equation $y=-7x+10$, and line joining A, and C has equation $y=2x+1$. We split the triangle along the vertical line $x=1$, and integrate first with respect to y. The desired integral is then

$$\begin{aligned}\int_D (2x+3y+1)\ \mathrm{d}x\mathrm{d}y &= \int_{-1}^{1}\left(\int_{-x-2}^{2x+1}(2x+3y+1)\mathrm{d}y\right)\mathrm{d}x \\ &\quad+\int_1^2\left(\int_{-x-2}^{-7x+10}(2x+3y+1)\mathrm{d}y\right)\mathrm{d}x \\ &= \int_{-1}^{1}\left(\frac{21}{2}x^2+9x-\frac{3}{2}\right)\mathrm{d}x \\ &\quad+\int_1^2\left(60x^2-198x+156\right)\mathrm{d}x \\ &= 4-1 \\ &= 3.\end{aligned}$$

3.5.27 Since f is positive and decreasing,

$$\int_0^1\int_0^1 f(x)f(y)(y-x)(f(x)-f(y))\mathrm{d}x\mathrm{d}y \geq 0,$$

from where the desired inequality follows.

3.5.28 The domain of integration is a triangle. The integral equals

$$\begin{aligned}\int_D xy(x+y)\mathrm{d}x\mathrm{d}y &= \int_0^1\left(\int_0^{1-x} xy(x+y)\ \mathrm{d}y\right)\ \mathrm{d}x \\ &= \int_0^1 x\left[x\frac{y^2}{2}+\frac{y^3}{3}\right]_0^{1-x}\ \mathrm{d}x \\ &= \int_0^1 x\left(\frac{x(1-x)^2}{2}+\frac{(1-x)^3}{3}\right)\ \mathrm{d}x \\ &= \frac{1}{30}.\end{aligned}$$

3.5.29 For $t\in[0;1]$, first argue that $\displaystyle\int_0^1 f(x)\mathrm{d}x \geq (1-t)f(t) \geq f(t)-t$. Hence

$$\int_0^1\int_0^1 (f(x)\mathrm{d}x)\ \mathrm{d}y \geq \int_0^1 (f\circ g)(y)\mathrm{d}y - \int_0^1 g(y)\mathrm{d}y.$$

Since $\displaystyle\int_0^1\int_0^1 f(x)\mathrm{d}x\mathrm{d}y = \int_0^1 f(x)\mathrm{d}x$, the desired inequality is established.

3.5.30 Put $f(x,y)=xy+y^2$. If I, II, III, IV stand for the intersection of the region with each quadrant, then

$$\int_{II} f(x,y)\mathrm{d}x\mathrm{d}y = \int_{II} f(-x,y)\mathrm{d}(-x)\mathrm{d}y = -\int_{I} f(x,y)\mathrm{d}x\mathrm{d}y,$$

$$\int_{IV} f(x,y)\mathrm{d}x\mathrm{d}y = \int_{IV} f(x,-y)\mathrm{d}x\mathrm{d}(-y) = -\int_{I} f(x,y)\mathrm{d}x\mathrm{d}y,$$

and

$$\int_{III} f(x,y)\mathrm{d}x\mathrm{d}y = \int_{III} f(-x,-y)\mathrm{d}(-x)\mathrm{d}(-y) = +\int_{I} f(x,y)\mathrm{d}x\mathrm{d}y.$$

Thus

$$\begin{aligned}\int_S (xy+y^2)\mathrm{d}x\mathrm{d}y &= \int_I f(x,y)\mathrm{d}x\mathrm{d}y + \int_{II} f(x,y)\mathrm{d}x\mathrm{d}y + \int_{III} f(x,y)\mathrm{d}x\mathrm{d}y + \int_{IV} f(x,y)\mathrm{d}x\mathrm{d}y \\ &= \int_I f(x,y)\mathrm{d}x\mathrm{d}y - \int_I f(x,y)\mathrm{d}x\mathrm{d}y + \int_I f(x,y)\mathrm{d}x\mathrm{d}y - \int_I f(x,y)\mathrm{d}x\mathrm{d}y \\ &= 0.\end{aligned}$$

3.5.31 We split the rectangle $[0;a]\times[0;b]$ into two triangles, depending on whether $bx < ay$ or $bx \geq ay$. Hence

$$\begin{aligned}\int_0^a\int_0^b e^{\max(b^2x^2,a^2y^2)}\,\mathrm{d}y\mathrm{d}x &= \int_{bx<ay} e^{\max(b^2x^2,a^2y^2)}\,\mathrm{d}y\mathrm{d}x + \int_{bx\geq ay} e^{\max(b^2x^2,a^2y^2)}\,\mathrm{d}y\mathrm{d}x \\ &= \int_{bx<ay} e^{a^2y^2}\,\mathrm{d}y\mathrm{d}x + \int_{bx\geq ay} e^{b^2x^2}\,\mathrm{d}y\mathrm{d}x \\ &= \int_0^b\int_0^{ay/b} e^{a^2y^2}\mathrm{d}x\mathrm{d}y + \int_0^a\int_0^{bx/a} e^{a^2y^2}\mathrm{d}y\mathrm{d}x \\ &= \int_0^b \frac{aye^{a^2y^2}}{b}\mathrm{d}y + \int_0^a \frac{bxe^{a^2y^2}}{a}\mathrm{d}x \\ &= \frac{e^{a^2b^2}-1}{2ab} + \frac{e^{a^2b^2}-1}{2ab} \\ &= \frac{e^{a^2b^2}-1}{ab}.\end{aligned}$$

3.5.32 Observe that $x \geq \frac{1}{2}(x+y)^2 \geq 0$. Hence we may take the positive square root giving $y \leq \sqrt{2x}-x$. Since $y \geq 0$, we must have $\sqrt{2x}-x \geq 0$ which means that $x \leq 2$. The integral equals

$$\begin{aligned}\int_0^2\left(\int_0^{\sqrt{2x}-x}\sqrt{xy}\mathrm{d}y\right)\mathrm{d}x &= \frac{2}{3}\int_0^2 \sqrt{x}(\sqrt{2x}-x)^{3/2}\mathrm{d}x \\ &= \frac{4}{3}\int_0^{\sqrt{2}} u^2(u\sqrt{2}-u^2)^{3/2}\mathrm{d}u \\ &= \frac{1}{6}\int_{-1}^1 (1-v^2)^{3/2}(1+v)^2\mathrm{d}v \\ &= \frac{1}{6}\int_{-\pi/2}^{\pi/2}\cos^4\theta(1+\sin^2\theta)\mathrm{d}\theta \\ &= \frac{7\pi}{96}.\end{aligned}$$

3.5.33 Observe that

$$\int_0^a \sin 2\pi x\;\mathrm{d}x = \begin{cases} 0 & \text{if } a \text{ is an integer} \\ \dfrac{1}{2\pi}(1-\cos 2\pi a) & \text{if } a \text{ is not an integer}\end{cases}$$

Thus

$$\int_0^a \sin 2\pi x\;\mathrm{d}x = 0 \iff a \text{ is an integer.}$$

Now

$$\sum_{k=1}^N \int_{R_k} \sin 2\pi x \sin 2\pi y\;\mathrm{d}x\mathrm{d}y = 0$$

since at least one of the sides of each R_k is an integer. Since

$$\int_R \sin 2\pi x \sin 2\pi y\;\mathrm{d}x\mathrm{d}y = \sum_{k=1}^N \int_{R_k} \sin 2\pi x \sin 2\pi y\;\mathrm{d}x\mathrm{d}y,$$

we deduce that at least one of the sides of R is an integer, finishing the proof.

3.5.34 We have

$$\int_0^1\int_0^1\cdots\int_0^1 (x_1x_2\cdots x_n)\mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_n = \prod_{k=1}^n\left(\int_0^1 x_k\mathrm{d}x_k\right) = \prod_{k=1}^n \frac{1}{2} = \frac{1}{2^n}.$$

3.5.35 This is

$$\begin{aligned}\int_0^1\int_0^1\cdots\int_0^1\left(\sum_{k=1}^n x_k\right)\mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_n &= \sum_{k=1}^n\int_0^1\int_0^1\cdots\int_0^1 x_k\mathrm{d}x_1\mathrm{d}x_2\ldots\mathrm{d}x_n\\ &= \sum_{k=1}^n\frac{1}{2}\\ &= \frac{n}{2}.\end{aligned}$$

3.5.41 The integral equals

$$\begin{aligned}\int_D \frac{1}{(x+y)^4}\mathrm{d}x\mathrm{d}y &= \int_1^3\left(\int_1^{4-x}\frac{\mathrm{d}y}{(x+y)^4}\,\mathrm{d}y\right)\,\mathrm{d}x\\ &= \int_1^3\left[-\frac{1}{3}(x+y)^{-3}\right]_1^{4-x}\,\mathrm{d}x\\ &= \frac{1}{3}\int_1^3\left(\frac{1}{(1+x)^3}-\frac{1}{64}\right)\,\mathrm{d}x\\ &= \frac{1}{48}.\end{aligned}$$

3.5.45 The integral equals

$$\begin{aligned}\int_D x\mathrm{d}x\mathrm{d}y &= \int_{-1}^{2/3}\left(\int_0^{x+1}\,\mathrm{d}y\right)x\,\mathrm{d}x + \int_{2/3}^4\left(\int_0^{2-\frac{x}{2}}\,\mathrm{d}y\right)x\,\mathrm{d}x\\ &= \int_{-1}^{2/3}x(x+1)\,\mathrm{d}x + \int_{2/3}^4 x\left(2-\frac{x}{2}\right)\,\mathrm{d}x\\ &= \frac{275}{54}.\end{aligned}$$

3.5.46 Make the change of variables $x_k = 1 - y_k$. Then

$$I = \int_0^1\int_0^1\cdots\int_0^1 \cos^2\left(\frac{\pi}{2n}(x_1+x_2+\cdots+x_n)\right)\mathrm{d}x_1\,\mathrm{d}x_2\ldots\,\mathrm{d}x_n$$

equals

$$\int_0^1\int_0^1\cdots\int_0^1 \sin^2\left(\frac{\pi}{2n}(y_1+y_2+\cdots+y_n)\right)\mathrm{d}y_1\,\mathrm{d}y_2\ldots\,\mathrm{d}y_n.$$

Since $\sin^2 t + \cos^2 t = 1$, we have $2I = 1$, and so $I = \dfrac{1}{2}$.

3.6.1 ❶ Put $x = \dfrac{u+v}{2}$ and $y = \dfrac{u-v}{2}$. Then $x+y = u$ and $x-y$. Observe that D' is the triangle in the uv plane bounded by the lines $u = 0, u = 1, v = u, v = -u$. Its image under Φ is the triangle bounded by the equations $x = 0, y = 0, x+y = 1$. Clearly also

$$\mathrm{d}x\wedge\mathrm{d}y = \frac{1}{2}\mathrm{d}u\wedge\mathrm{d}v.$$

❷ From the above

$$\begin{aligned}\int_D (x+y)^2e^{x^2-y^2}\mathrm{d}A &= \frac{1}{2}\int_{D'} u^2e^{uv}\mathrm{d}u\mathrm{d}v\\ &= \frac{1}{2}\int_0^1\int_{-u}^u u^2e^{uv}\mathrm{d}u\mathrm{d}v\\ &= \frac{1}{2}\int_0^1 u(e^{u^2}-e^{-u^2})\mathrm{d}u\\ &= \frac{1}{4}(e+e^{-1}-2).\end{aligned}$$

3.6.4 Here we argue that

$$\mathrm{d}u = y\mathrm{d}x + x\mathrm{d}y,$$

$$\mathrm{d}v = -2x\mathrm{d}x + 2y\mathrm{d}y.$$

Taking the wedge product of differential forms,

$$\mathrm{d}u \wedge \mathrm{d}v = 2(y^2 + x^2)\mathrm{d}x \wedge \mathrm{d}y.$$

Hence

$$\begin{aligned} f(x,y)\mathrm{d}x \wedge \mathrm{d}y &= (y^4 - x^4)\frac{1}{2(y^2+x^2)}\mathrm{d}u \wedge \mathrm{d}v \\ &= \frac{1}{2}(y^2 - x^2)\mathrm{d}u \wedge \mathrm{d}v \\ &= \frac{v}{2}\mathrm{d}u \wedge \mathrm{d}v \end{aligned}$$

The region transforms into

$$\Delta = [a;b] \times [0;1].$$

The integral becomes

$$\begin{aligned} \int_D f(x,y)\mathrm{d}x \wedge \mathrm{d}y &= \int_\Delta v \ \mathrm{d}u \wedge \mathrm{d}v \\ &= \frac{1}{2}\left(\int_a^b \ \mathrm{d}u\right)\left(\int_0^1 v \ \mathrm{d}v\right) \\ &= \frac{b-a}{4}. \end{aligned}$$

3.6.5 ❶ Formally,

$$\begin{aligned} \int_0^1\int_0^1 \frac{\mathrm{d}x\mathrm{d}y}{1-xy} &= \int_0^1\int_0^1 (1 + xy + x^2y^2 + x^3y^3 + \cdots)\mathrm{d}x\mathrm{d}y \\ &= \int_0^1 \left(y + \frac{xy^2}{2} + \frac{x^2y^3}{3} + \frac{x^3y^4}{4} + \cdots\right)_0^1 \mathrm{d}x \\ &= \int_0^1 (1 + \frac{x}{2} + \frac{x^2}{3} + \frac{x^3}{4} + \cdots)\mathrm{d}x \\ &= 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots \end{aligned}$$

❷ This change of variables transforms the square $[0;1] \times [0;1]$ in the xy plane into the square with vertices at $(0,0)$, $(1,1)$, $(2,0)$, and $(1,-1)$ in the uv plane. We will split this region of integration into two disjoint triangles: T_1 with vertices at $(0,0)$, $(1,1)$, $(1,-1)$, and T_2 with vertices at $(1,-1)$, $(1,1)$, $(2,0)$. Observe that

$$\mathrm{d}x \wedge \mathrm{d}y = \frac{1}{2}\mathrm{d}u \wedge \mathrm{d}v,$$

and that $u + v = 2x, u - v = 2y$ and so $4xy = u^2 - v^2$. The integral becomes

$$\begin{aligned} \int_0^1\int_0^1 \frac{\mathrm{d}x\mathrm{d}y}{1-xy} &= \frac{1}{2}\int_{T_1 \cup T_2} \frac{\mathrm{d}u \wedge \mathrm{d}v}{1 - \frac{u^2-v^2}{4}} \\ &= 2\int_0^1 \left(\int_{-u}^{u} \frac{\mathrm{d}v}{4 - u^2 + v^2}\right)\mathrm{d}u + 2\int_1^2 \left(\int_{u-2}^{2-u} \frac{\mathrm{d}v}{4 - u^2 + v^2}\right)\mathrm{d}u, \end{aligned}$$

as desired.

❸ This follows by using the identity

$$\int_0^t \frac{\mathrm{d}\omega}{1+\Omega^2} = \arctan t.$$

❹ This is straightforward but tedious!

3.7.1 The integral in Cartesian coordinates is

$$\begin{aligned} \int_1^{\sqrt{15}}\int_1^{\sqrt{16-y^2}} xy\mathrm{d}x\mathrm{d}y &= \frac{1}{2}\int\int_1^{\sqrt{15}} 15y - y^3\mathrm{d}y \\ &= \frac{49}{2}. \end{aligned}$$

The integral in polar coordinates is

$$\begin{aligned}
\int_{\arcsin\frac{1}{4}}^{\frac{\pi}{4}}\int_{1/\sin\theta}^{4} r^3\sin\theta\cos\theta\mathrm{d}r\mathrm{d}\theta+\int_{\frac{\pi}{4}}^{\arccos\frac{1}{4}}\int_{1/\cos\theta}^{4} r^3\sin\theta\cos\theta\mathrm{d}r\mathrm{d}\theta &= \frac{1}{4}\int_{\arcsin\frac{1}{4}}^{\frac{\pi}{4}}\left(4^4-\frac{1}{\sin^4\theta}\right)\sin\theta\cos\theta\mathrm{d}\theta \\
&\quad +\frac{1}{4}\int_{\frac{\pi}{4}}^{\arccos\frac{1}{4}}\left(4^4-\frac{1}{\cos^4\theta}\right)\sin\theta\cos\theta\mathrm{d}\theta \\
&= \frac{4^4}{4}\int_{\arcsin\frac{\pi}{4}}^{\arccos\frac{1}{4}}\sin\theta\cos\theta\mathrm{d}\theta \\
&\quad -\frac{1}{4}\int_{\arcsin\frac{1}{4}}^{\frac{\pi}{4}}(\cot\theta)(\csc^2\theta)\mathrm{d}\theta \\
&\quad -\frac{1}{4}\int_{\frac{\pi}{4}}^{\arccos\frac{1}{4}}(\tan\theta)(\sec^2\theta)\mathrm{d}\theta \\
&= 28-\frac{7}{4}-\frac{7}{4} \\
&= \frac{49}{2}
\end{aligned}$$

3.7.2 Using polar coordinates,

$$\begin{aligned}
\int_D x^2-y^2\mathrm{d}x\mathrm{d}y &= \int_{-\pi/2}^{\pi/2}\left(\int_0^{2\cos\theta}\rho^3\,\mathrm{d}\rho\right)(\cos^2\theta-\sin^2\theta)\mathrm{d}\theta \\
&= 8\int_0^{\pi/2}\cos^4\theta(\cos^2\theta-\sin^2\theta)\,\mathrm{d}\theta \\
&= \pi.
\end{aligned}$$

3.7.3 Using polar coordinates,

$$\begin{aligned}
\int_D\sqrt{xy}\mathrm{d}x\mathrm{d}y &= 4\int_0^{\pi/4}\left(\int_0^{\sqrt{\sin 2\theta}}\rho\sqrt{\rho^2\cos\theta\sin\theta}\,\mathrm{d}\rho\right)\mathrm{d}\theta \\
&= \frac{4}{3}\int_0^{\pi/4}(\sqrt{\sin 2\theta})^3\sqrt{\cos\theta\sin\theta}\,\mathrm{d}\theta \\
&= \frac{4}{3\sqrt{2}}\int_0^{\pi/4}\sin^2 2\theta\,\mathrm{d}\theta \\
&= \frac{\pi\sqrt{2}}{12}.
\end{aligned}$$

3.7.4 Using $x=a\rho\cos\theta$, $y=b\rho\sin\theta$, the integral becomes

$$(ab)\left(\int_0^{2\pi}a^3\cos^3\theta+b^3\sin^3\theta\mathrm{d}\theta\right)\left(\int_0^1\rho^4\mathrm{d}\rho\right)=\frac{2}{15}(ab)(a^3+b^3).$$

3.7.8 Using polar coordinates,

$$\begin{aligned}
\int_D f(x,y)\mathrm{d}A &= \int_0^{\pi/6}\left(\int_{2\sin\theta}^{1}\rho^2\,\mathrm{d}\rho\right)\mathrm{d}\theta \\
&= \frac{1}{3}\int_0^{\pi/6}(1-8\sin^3\theta)\,\mathrm{d}\theta \\
&= \frac{\pi}{18}-\frac{16}{9}+\sqrt{3}.
\end{aligned}$$

3.7.9 Using polar coordinates the integral becomes

$$\int_0^{\pi/2}\left(\int_0^{2\cos\theta}\rho^4\mathrm{d}\rho\right)\cos^2\theta\sin\theta\mathrm{d}\theta=\frac{4}{5}.$$

3.7.11 Using polar coordinates the integral becomes

$$\int_{-\pi/4}^{\pi/4}\left(\int_{1/\cos\theta}^{2\cos\theta}\frac{1}{\rho^3}\mathrm{d}\rho\right)\mathrm{d}\theta=\int_0^{\pi/4}\left(\cos^2\theta-\frac{\sec^2\theta}{4}\right)\mathrm{d}\theta=\frac{\pi}{8}.$$

3.7.12 Put
$$D' = \{(x, y) \in \mathbb{R}^2 : y \geq x, x^2 + y^2 - y \leq 0, x^2 + y^2 - x \leq 0\}.$$
Then the integral equals
$$2 \int_{D'} (x + y)^2 \mathrm{d}x\mathrm{d}y.$$
Using polar coordinates the integral equals
$$\begin{aligned} 2 \int_{\pi/4}^{\pi/2} (\cos\theta + \sin\theta)^2 \left(\int_0^{\cos\theta} \rho^3 \mathrm{d}\rho\right) \mathrm{d}\theta &= \frac{1}{2}\int_{\pi/4}^{\pi/2} \cos^4\theta(1 + 2\sin\theta\cos\theta)\mathrm{d}\theta \\ &= \frac{3\pi}{64} - \frac{5}{48}. \end{aligned}$$

3.7.13 Observe that $D = D_2 \setminus D_1$ where D_2 is the disk limited by the equation $x^2 + y^2 = 1$ and D_1 is the disk limited by the equation $x^2 + y^2 = y$. Hence
$$\int_D \frac{\mathrm{d}x\mathrm{d}y}{(1 + x^2 + y^2)^2} = \int_{D_2} \frac{\mathrm{d}x\mathrm{d}y}{(1 + x^2 + y^2)^2} - \int_{D_1} \frac{\mathrm{d}x\mathrm{d}y}{(1 + x^2 + y^2)^2}.$$
Using polar coordinates we have
$$\int_{D_2} \frac{\mathrm{d}x\mathrm{d}y}{(1 + x^2 + y^2)^2} = \int_0^{2\pi}\int_0^1 \frac{\rho}{(1 + \rho^2)^2}\mathrm{d}\rho\mathrm{d}\theta = \frac{\pi}{2}$$
and
$$\begin{aligned} \int_{D_1} \frac{\mathrm{d}x\mathrm{d}y}{(1 + x^2 + y^2)^2} &= 2\int_0^{\pi/2}\int_0^{\sin\theta} \frac{\rho}{(1 + \rho^2)^2}\mathrm{d}\rho\mathrm{d}\theta = \int_0^{\pi/2} \frac{\sin^2\theta\mathrm{d}\theta}{1 + \sin^2\theta} \\ &= \int_0^{+\infty} \frac{\mathrm{d}t}{t^2 + 1} - \frac{\mathrm{d}t}{2t^2 + 1} = \frac{\pi}{2} - \frac{\pi\sqrt{2}}{4}. \end{aligned}$$
(We evaluated this last integral using $t = \tan\theta$) Finally, the integral equals
$$\frac{\pi}{2} - \left(\frac{\pi}{2} - \frac{\pi\sqrt{2}}{4}\right) = \frac{\pi\sqrt{2}}{4}.$$

3.7.14 We have
$$2x\mathrm{d}x = \cos\theta\mathrm{d}\rho - \rho\sin\theta\mathrm{d}\theta, \quad 2y\mathrm{d}y = \sin\theta\mathrm{d}\rho + \rho\cos\theta\mathrm{d}\theta,$$
whence
$$4xy\mathrm{d}x \wedge \mathrm{d}y = \rho\mathrm{d}\rho \wedge \mathrm{d}\theta.$$
It follows that
$$\begin{aligned} x^3y^3\sqrt{1 - x^4 - y^4}\,\mathrm{d}x \wedge \mathrm{d}y &= \frac{1}{4}(x^2y^2)(\sqrt{1 - x^4 - y^4})(4xy\,\mathrm{d}x \wedge \mathrm{d}y) \\ &= \frac{1}{4}(\rho^3\cos\theta\sin\theta\sqrt{1 - \rho^2})\mathrm{d}\rho \wedge \mathrm{d}\theta \end{aligned}$$
Observe that
$$x^4 + y^4 \leq 1 \implies \rho^2\cos^2\theta + \rho^2\sin^2\theta \leq 1 \implies \rho \leq 1.$$
Since the integration takes place on the first quadrant, we have $0 \leq \theta \leq \pi/2$. Hence the integral becomes
$$\begin{aligned} \int_0^{\pi/2}\int_0^1 \frac{1}{4}(\rho^3\cos\theta\sin\theta\sqrt{1 - \rho^2})\mathrm{d}\rho\mathrm{d}\theta &= \frac{1}{4}\left(\int_0^{\pi/2}\cos\theta\sin\theta\mathrm{d}\theta\right)\left(\int_0^1 \rho^3\sqrt{1 - \rho^2}\mathrm{d}\rho\right) \\ &= \frac{1}{4}\cdot\frac{1}{2}\cdot\frac{2}{15} \\ &= \frac{1}{60}. \end{aligned}$$

3.7.15 ❶ Using polar coordinates
$$I_a = \int_0^{2\pi}\left(\int_0^a \rho e^{-\rho^2}\mathrm{d}\rho\right)\mathrm{d}\theta = \pi(1 - e^{-a^2}).$$

❷ The domain of integration of J_a is a square of side $2a$ centred at the origin. The respective domains of integration of I_a and $I_{a\sqrt{2}}$ are the inscribed and the exscribed circles to the square.

❸ First observe that
$$J_a = \left(\int_{-a}^a e^{-x^2}\mathrm{d}x\right)^2.$$
Since both I_a and $I_{a\sqrt{2}}$ tend to π as $a \to +\infty$, we deduce that $J_a \to \pi$. This gives the result.

3.7.16

$$\begin{aligned}
\int_{4\leq x^2+y^2\leq 16} \frac{1}{x^2+xy+y^2}\mathrm{d}A &= \int_0^{2\pi}\int_2^4 \frac{r}{r^2+r^2\sin\theta\cos\theta}\mathrm{d}r\mathrm{d}\theta \\
&= \int_0^{2\pi}\int_2^4 \frac{1}{r(1+\sin\theta\cos\theta)}\mathrm{d}r\mathrm{d}\theta \\
&= \left(\int_0^{2\pi}\frac{\mathrm{d}\theta}{1+\sin\theta\cos\theta}\right)\left(\int_2^4\frac{\mathrm{d}r}{r}\right) \\
&= \left(\int_0^{2\pi}\frac{\mathrm{d}\theta}{1+\sin\theta\cos\theta}\right)\log 2 \\
&= 2\left(\int_0^{2\pi}\frac{\mathrm{d}\theta}{2+\sin 2\theta}\right)\log 2 \\
&= 4\left(\int_0^{\pi}\frac{\mathrm{d}\theta}{2+\sin 2\theta}\right)\log 2 \\
&= 4I\log 2,
\end{aligned}$$

so the problem reduces to evaluate $I = \int_0^{\pi}\frac{\mathrm{d}\theta}{2+\sin 2\theta}$. To find this integral, we now use what has been dubbed as "the world's sneakiest substitution"[1]: we put $\tan\theta = t$. In so doing we have to pay attention to the fact that $\theta\mapsto\tan\theta$ is not continuous on $[0;\pi]$, so we split the interval of integration into two pieces, $[0;\pi] = [0;\frac{\pi}{2}]\cup]\frac{\pi}{2};\pi]$. Then $\sin 2\theta = \frac{2t}{1+t^2}$, $\cos 2\theta = \frac{1-t^2}{1+t^2}$, $\mathrm{d}\theta = \frac{\mathrm{d}t}{1+t^2}$. Hence

$$\begin{aligned}
\int_0^{\pi}\frac{\mathrm{d}\theta}{2+\sin 2\theta} &= \int_0^{\pi/2}\frac{\mathrm{d}\theta}{2+\sin 2\theta} + \int_{\pi/2}^{\pi}\frac{\mathrm{d}\theta}{2+\sin 2\theta} \\
&= \int_0^{+\infty}\frac{\frac{\mathrm{d}t}{1+t^2}}{2+\frac{2t}{1+t^2}} + \int_{-\infty}^{0}\frac{\frac{\mathrm{d}t}{1+t^2}}{2+\frac{2t}{1+t^2}} \\
&= \int_0^{+\infty}\frac{\mathrm{d}t}{2(t^2+t+1)} + \int_{-\infty}^{0}\frac{\mathrm{d}t}{2(t^2+t+1)} \\
&= \frac{2}{3}\int_0^{+\infty}\frac{\mathrm{d}t}{(\frac{2t}{\sqrt{3}}+\frac{1}{\sqrt{3}})^2+1} + \frac{2}{3}\int_{-\infty}^{0}\frac{\mathrm{d}t}{(\frac{2t}{\sqrt{3}}+\frac{1}{\sqrt{3}})^2+1} \\
&= \frac{\sqrt{3}}{3}\Big|_0^{+\infty}\arctan\left(\frac{2t\sqrt{3}}{3}+\frac{\sqrt{3}}{3}\right) + \frac{\sqrt{3}}{3}\Big|_{-\infty}^{0}\arctan\left(\frac{2t\sqrt{3}}{3}+\frac{\sqrt{3}}{3}\right) \\
&= \frac{\sqrt{3}}{3}\left(\frac{\pi}{2}-\frac{\pi}{6}\right) + \frac{\sqrt{3}}{3}\left(\frac{\pi}{6}-\left(-\frac{\pi}{2}\right)\right) \\
&= \frac{\pi\sqrt{3}}{3}.
\end{aligned}$$

We conclude that

$$\int_{4\leq x^2+y^2\leq 16}\frac{1}{x^2+xy+y^2}\mathrm{d}A = \frac{4\pi\sqrt{3}\log 2}{3}.$$

3.7.17 Recall from formula 1.14 that the area enclosed by a simple closed curve Γ is given by

$$\frac{1}{2}\int_\Gamma x\mathrm{d}y - y\mathrm{d}x.$$

Using polar coordinates

$$\begin{aligned}
x\mathrm{d}y - y\mathrm{d}x &= (\rho\cos\theta)(\sin\theta\mathrm{d}\rho + \rho\cos\theta\mathrm{d}\theta) - (\rho\sin\theta)(\cos\theta\mathrm{d}\rho - \rho\sin\theta\mathrm{d}\theta) \\
&= \rho^2\mathrm{d}\theta.
\end{aligned}$$

Parametrise the curve enclosing the region by polar coordinates so that the region is tangent to the polar axis at the origin. Let the equation of the curve be $\rho = f(\theta)$. The area of the region is then given by

$$\frac{1}{2}\int_0^{\pi}\rho^2\mathrm{d}\theta = \frac{1}{2}\int_0^{\pi}(f(\theta))^2\mathrm{d}\theta = \frac{1}{2}\int_0^{\pi/2}((f(\theta))^2 + (f(\theta+\pi/2))^2)\mathrm{d}\theta.$$

By the Pythagorean Theorem, the integral above is the integral of the square of the chord in question. If no two points are farther than 2 units, their squares are no farther than 4 units, and so the area

$$< \frac{1}{2}\int_0^{\pi/2}4\mathrm{d}\theta = \pi,$$

[1] by Michael Spivak, whose *Calculus* book I recommend greatly.

a contradiction.

3.7.18 Let $I(S)$ denote the integral sought over a region S. Since $D(x, y) = 0$ inside R, $I(R) = A$. Let $\mathscr{L}$ be a side of R with length l and let $S(\mathscr{L})$ be the half strip consisting of the points of the plane having a point on $\mathscr{L}$ as nearest point of R. Set up coordinates uv so that u is measured parallel to $\mathscr{L}$ and v is measured perpendicular to L. Then

$$I(S(\mathscr{L})) = \int_0^l \int_0^{+\infty} e^{-v}\,\mathrm{d}u\mathrm{d}v = l.$$

The sum of these integrals over all the sides of R is L.

If $\mathscr{V}$ is a vertex of R, the points that have $\mathscr{V}$ as nearest from R lie inside an angle $S(\mathscr{V})$ bounded by the rays from $\mathscr{V}$ perpendicular to the edges meeting at $\mathscr{V}$. If α is the measure of that angle, then using polar coordinates

$$I(S(\mathscr{V})) = \int_0^\alpha \int_0^{+\infty} \rho e^{-\rho}\,\mathrm{d}\rho\mathrm{d}\theta = \alpha.$$

The sum of these integrals over all the vertices of R is 2π. Assembling all these integrals we deduce the result.

3.8.1 We have

$$\begin{aligned}
\int_E z\mathrm{d}V &= \int_0^1 \int_0^{1-y} \int_0^{1-z} z\mathrm{d}x\mathrm{d}z\mathrm{d}y \\
&= \int_0^1 \int_0^{1-y} z - z^2\mathrm{d}z\mathrm{d}y \\
&= \int_0^1 \frac{(1-y)^2}{2} - \frac{(1-y)^3}{3}\mathrm{d}y \\
&= \frac{(1-y)^4}{12} - \frac{(1-y)^3}{6}\Big|_0^1 \\
&= = \frac{1}{6}.
\end{aligned}$$

3.8.3 Let $A = (1, 1, 1)$, $B = (1, 0, 0)$, $C = (0, 0, 1)$, and $O = (0, 0, 0)$. We have four planes passing through each triplet of points:

$$\begin{array}{lll}
P_1: & A, B, C, & x - y + z = 1 \\
P_2: & A, B, O & z = y \\
P_3: & A, C, O & x = y \\
P_4: & B, C, O & y = 0.
\end{array}$$

Using the order of integration $\mathrm{d}z\mathrm{d}x\mathrm{d}y$, z sweeps from P_2 to P_1, so the limits are $z = y$ to $z = 1 - x + y$. The projection of the solid on the xy plane produces the region bounded by the lines $x = 0$, $x = 1$ and $x = y$ on the first quadrant of the xy-plane. Thus

$$\begin{aligned}
\int_0^1 \int_0^x \int_y^{1-x+y} \mathrm{d}z\mathrm{d}y\mathrm{d}x &= \int_0^1 \int_0^x (1-x)\mathrm{d}y\mathrm{d}x \\
&= \int_0^1 \left(x - x^2\right)\mathrm{d}x \\
&= \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1 \\
&= \frac{1}{6}.
\end{aligned}$$

We use the same limits of integration as in the previous integral. We have

$$\begin{aligned}
\int_0^1 \int_0^x \int_y^{1-x+y} x\mathrm{d}z\mathrm{d}y\mathrm{d}x &= \int_0^1 \int_0^x (x - x^2)\mathrm{d}y\mathrm{d}x \\
&= \int_0^1 \left(x^2 - x^3\right)\mathrm{d}x \\
&= \left(\frac{x^3}{3} - \frac{x^4}{4}\right)\Big|_0^1 \\
&= \frac{1}{12}.
\end{aligned}$$

3.8.4 We have

$$\int_E x\mathrm{d}V = \int_0^3 \int_0^{\sqrt{9-x^2}} \int_0^{y/3} x\mathrm{d}z\mathrm{d}y\mathrm{d}x = \frac{27}{8}.$$

3.8.5 The desired integral is

$$\begin{aligned}
\int_0^1\int_0^1\int_0^{\infty}\frac{\mathrm{d}x\mathrm{d}y\mathrm{d}z}{(1+x^2z^2)(1+y^2z^2)} &= \int_0^1\int_0^1\int_0^{\infty}\frac{1}{x^2-y^2}\left(\frac{x^2}{1+x^2z^2}-\frac{y^2}{1+y^2z^2}\right)\mathrm{d}x\mathrm{d}y\mathrm{d}z\\
&= \int_0^1\int_0^1\frac{1}{x^2-y^2}\left(x\arctan(xz)-y\arctan(yz)\right)\Big|_{z=0}^{z=\infty}\mathrm{d}x\mathrm{d}y\\
&= \int_0^1\int_0^1\frac{\pi(x-y)}{2(x^2-y^2)}\mathrm{d}x\mathrm{d}y\\
&= \int_0^1\int_0^1\frac{\pi}{2(x+y)}\mathrm{d}x\mathrm{d}y\\
&= \frac{\pi}{2}\int_0^1\log(y+1)-\log y\mathrm{d}y\\
&= \frac{\pi}{2}\cdot\left((y+1)\log(y+1)-(y+1)-y\log y+y\right)\Big|_0^1\\
&= \pi\log 2.
\end{aligned}$$

3.9.1 Cartesian:

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}}\int_{x^2+y^2}^{\sqrt{x^2+y^2}}\mathrm{d}z\mathrm{d}x\mathrm{d}y.$$

Cylindrical:

$$\int_0^1\int_0^{2\pi}\int_{r^2}^{r}r\mathrm{d}z\mathrm{d}\theta\mathrm{d}r.$$

Spherical:

$$\int_{\pi/4}^{\pi/2}\int_0^{2\pi}\int_0^{(\cos\phi)/(\sin\phi)^2}r^2\sin\phi\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi.$$

The volume is $\dfrac{\pi}{3}$.

3.9.2

1. Since $x^2+y^2\leq z\leq\sqrt{4-x^2-y^2}$, we start our integration with the z-variable. Observe that if (x,y,z) is on the intersection of the surfaces then

$$z^2+z=4 \implies z=\frac{-1\pm\sqrt{17}}{2}.$$

Since $x^2+y^2+z^2=4 \implies -2\leq z\leq 2$, we must have $z=\dfrac{\sqrt{17}-1}{2}$ only. The projection of the circle of intersection of the paraboloid and the sphere onto the xy-plane satisfies the equation

$$z^2+z=4 \implies x^2+y^2+(x^2+y^2)^2=4 \implies x^2+y^2=\frac{\sqrt{17}-1}{2},$$

a circle of radius $\sqrt{\dfrac{\sqrt{17}-1}{2}}$. The desired integral is thus

$$\int_{-\sqrt{\frac{\sqrt{17}-1}{2}}}^{\sqrt{\frac{\sqrt{17}-1}{2}}}\int_{-\sqrt{\frac{\sqrt{17}-1}{2}-x^2}}^{\sqrt{\frac{\sqrt{17}-1}{2}-x^2}}\int_{x^2+y^2}^{\sqrt{4-x^2-y^2}}x\mathrm{d}z\mathrm{d}y\mathrm{d}x.$$

2. The z-limits remain the same as in the Cartesian coordinates, but translated into cylindrical coordinates, and so $r^2\leq z\leq\sqrt{4-r^2}$. The projection of the intersection circle onto the xy-plane is again a circle with centre at the origin and radius $\sqrt{\dfrac{\sqrt{17}-1}{2}}$. The desired integral

$$\int_0^{2\pi}\int_0^{\sqrt{\frac{\sqrt{17}-1}{2}}}\int_{r^2}^{\sqrt{4-r^2}}r^2\cos\theta\mathrm{d}z\mathrm{d}r\mathrm{d}\theta.$$

3. Observe that

$$z = x^2 + y^2 \implies r\cos\phi = r^2(\cos\theta)^2(\sin\phi)^2 + r^2(\sin\theta)^2(\sin\phi)^2 \implies r \in \{0, (\csc\phi)(\cot\phi)\}.$$

It is clear that the limits of the angle θ are from $\theta = 0$ to $\theta = 2\pi$. The angle ϕ starts at $\phi = 0$. Now,

$$z = r\cos\phi \implies \cos\phi = \frac{\frac{\sqrt{17}-1}{2}}{2} \implies \phi = \arccos\left(\frac{\sqrt{17}-1}{4}\right)$$

The desired integral

$$\int_0^{2\pi}\int_0^{\arccos\left(\frac{\sqrt{17}-1}{4}\right)}\int_{(\csc\phi)(\cot\phi)}^{2} r^3\cos\theta\sin^2\phi \, dr d\phi d\theta.$$

Perhaps it is easiest to evaluate the integral using cylindrical coordinates. We obtain

$$\int_0^{2\pi}\int_0^{\sqrt{\frac{\sqrt{17}-1}{2}}}\int_{r^2}^{\sqrt{4-r^2}} r^2\cos\theta \, dz dr d\theta = 0,$$

a conclusion that is easily reached, since the integrand is an odd function of x and the domain of integration is symmetric about the origin in x.

3.9.3 Cartesian:

$$\int_{-\sqrt{3}}^{\sqrt{3}}\int_{-\sqrt{3-y^2}}^{\sqrt{3-y^2}}\int_1^{\sqrt{4-x^2-y^2}} dz dx dy.$$

Cylindrical:

$$\int_0^{\sqrt{3}}\int_0^{2\pi}\int_1^{\sqrt{4-r^2}} r \, dz d\theta dr.$$

Spherical:

$$\int_0^{\pi/3}\int_0^{2\pi}\int_{1/\cos\phi}^{2} r^2\sin\phi \, dr d\theta d\phi.$$

The volume is $\frac{5\pi}{3}$.

3.9.5 We have

$$\int_E y \, dV = \int_0^{2\pi}\int_1^2\int_0^{2+r\cos\theta} r^2\sin\theta \, dz dr d\theta = 0.$$

3.9.7 $\frac{\pi}{96}$

3.9.8 $\frac{\pi}{14}$

3.9.9 We put

$$x = \rho\cos\theta\sin\phi\sin t; \; y = \rho\sin\theta\sin\phi\sin t; \; u = \rho\cos\phi\sin t; \; v = \rho\cos t.$$

Upon using $\sin^2 a + \cos^2 a = 1$ three times,

$$\begin{aligned} x^2+y^2+u^2+v^2 &= r^2\cos^2\theta\sin^2\phi\sin^2 t + r^2\sin^2\theta\sin^2\phi\sin^2 t + r^2\cos^2\phi\sin^2 t + r^2\cos^2 t \\ &= r^2\cos^2\theta\sin^2\phi + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\phi \\ &= r^2\cos^2\theta + r^2\sin^2\theta \\ &= r^2. \end{aligned}$$

Now,

$$\begin{aligned} dx &= \cos\theta\sin\phi\sin t dr - \rho\sin\theta\sin\phi\sin t d\theta + \rho\cos\theta\cos\phi\sin t d\phi + \rho\cos\theta\sin\phi\cos t dt \\ dy &= \sin\theta\sin\phi\sin t dr + \rho\cos\theta\sin\phi\sin t d\theta + \rho\sin\theta\cos\phi\sin t d\phi + \rho\sin\theta\sin\phi\cos t dt \\ du &= \cos\phi\sin t dr - \rho\sin\phi\sin t d\phi + \rho\cos\phi\cos t dt \\ dv &= \cos t dr - \rho\sin t dt \end{aligned}.$$

After some calculation,

$$dx \wedge dy \wedge du \wedge dv = r^3 \sin\phi \sin^2 t dr \wedge d\phi \wedge d\theta \wedge dt.$$

Therefore

$$\begin{aligned}\iiiint_{x^2+y^2+u^2+v^2\leq 1} e^{x^2+y^2+u^2+v^2} dxdydudv &= \int_0^{\pi}\int_0^{2\pi}\int_0^{\pi}\int_0^1 r^3 e^{r^2} \sin\phi \sin^2 t dr d\phi d\theta dt \\ &= \left(\int_0^1 r^3 e^{r^2} dr\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\pi} \sin\phi d\phi\right)\left(\int_0^{\pi} \sin^2 t dt\right) \\ &= \left(\frac{1}{2}\right)(2\pi)(2)\left(\frac{\pi}{2}\right) \\ &= \pi^2.\end{aligned}$$

3.9.10 We make the change of variables

$$u = x+y+z \implies du = dx+dy+dz,$$

$$uv = y+z \implies udv + vdu = dy+dz,$$

$$uvw = z \implies uvdw + uwdv + vwdu = dz.$$

This gives

$$x = u(1-v),$$
$$y = uv(1-w),$$
$$z = uvw,$$
$$u^2v\, du \wedge dv \wedge dw = dx \wedge dy \wedge dz.$$

To find the limits of integration we observe that the limits of integration using $dx \wedge dy \wedge dz$ are

$$0 \leq z \leq 1,\ 0 \leq y \leq 1-z,\ 0 \leq x \leq 1-y-z.$$

This translates into

$$0 \leq uvw \leq 1,\ 0 \leq uv - uvw \leq 1 - uvw,\ 0 \leq u - uv \leq 1 - uv + uvw - uvw.$$

Thus

$$0 \leq uvw \leq 1,\ 0 \leq uv \leq 1,\ 0 \leq u \leq 1,$$

which finally give

$$0 \leq u \leq 1,\ 0 \leq v \leq 1,\ 0 \leq w \leq 1.$$

The integral sought is then, using the fact that for positive integers m, n one has

$$\int_0^1 x^m(1-x)^n\, dx = \frac{m!n!}{(m+n+1)!},$$

we deduce,

$$\int_0^1\int_0^1\int_0^1 u^{20}v^{18}w^8(1-u)^4(1-v)(1-w)^9\, dudvdw,$$

which in turn is

$$\left(\int_0^1 u^{20}(1-u)^4 du\right)\left(\int_0^1 v^{18}(1-v)dv\right)\left(\int_0^1 w^8(1-w)^9 dw\right) = \frac{1}{265650}\cdot\frac{1}{380}\cdot\frac{1}{437580}$$

which is

$$= \frac{1}{44172388260000}.$$

3.10.1 We parametrise the surface by letting $x = u, y = v, z = u + v^2$. Observe that the domain D of Σ is the square $[0;1] \times [0;2]$. Observe that

$$dx \wedge dy = du \wedge dv,$$

$$dy \wedge dz = -du \wedge dv,$$

$$dz \wedge dx = -2vdu \wedge dv,$$

and so

$$\left|\left|d^2x\right|\right| = \sqrt{2+4v^2}du \wedge dv.$$

The integral becomes

$$\begin{aligned}\int_{\Sigma} y\left|\left|d^2\mathbf{x}\right|\right| &= \int_0^2\int_0^1 v\sqrt{2+4v^2}\,du\,dv\\ &= \left(\int_0^1 du\right)\left(\int_0^2 y\sqrt{2+4v^2}\,dv\right)\\ &= \frac{13\sqrt{2}}{3}.\end{aligned}$$

3.10.2 Using $x = r\cos\theta$, $y = r\sin\theta$, $1 \le r \le 2$, $0 \le \theta \le 2\pi$, the surface area is

$$\sqrt{2}\int_0^{2\pi}\int_1^2 r\,dr\,d\theta = 3\pi\sqrt{2}.$$

3.10.3 We use spherical coordinates, $(x, y, z) = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$. Here $\theta \in [0; 2\pi]$ is the latitude and $\phi \in [0; \pi]$ is the longitude. Observe that

$$dx \wedge dy = \sin\phi\cos\phi\,d\phi \wedge d\theta,$$

$$dy \wedge dz = \cos\theta\sin^2\phi\,d\phi \wedge d\theta,$$

$$dz \wedge dx = -\sin\theta\sin^2\phi\,d\phi \wedge d\theta,$$

and so

$$\left|\left|d^2\mathbf{x}\right|\right| = \sin\phi\,d\phi \wedge d\theta.$$

The integral becomes

$$\begin{aligned}\int_{\Sigma} x^2\left|\left|d^2\mathbf{x}\right|\right| &= \int_0^{2\pi}\int_0^{\pi}\cos^2\theta\sin^3\phi\,d\phi\,d\theta\\ &= \frac{4\pi}{3}.\end{aligned}$$

3.10.4 Put $x = u, y = v, z^2 = u^2 + v^2$. Then

$$dx = du,\ dy = dv,\ z\,dz = u\,du + v\,dv,$$

whence

$$dx \wedge dy = du \wedge dv, dy \wedge dz = -\frac{u}{z}du \wedge dv, dz \wedge dx = -\frac{v}{z}du \wedge dv,$$

and so

$$\begin{aligned}\left|\left|d^2\mathbf{x}\right|\right| &= \sqrt{(dx \wedge dy)^2 + (dz \wedge dx)^2 + (dy \wedge dz)^2}\\ &= \sqrt{1 + \frac{u^2+v^2}{z^2}}\,du \wedge dv\\ &= \sqrt{2}\,du \wedge dv.\end{aligned}$$

Hence

$$\int_{\Sigma} z\left|\left|d^2\mathbf{x}\right|\right| = \int_{u^2+v^2\le 1}\sqrt{u^2+v^2}\,\sqrt{2}\,du\,dv = \sqrt{2}\int_0^{2\pi}\int_0^1\rho^2\,d\rho\,d\theta = \frac{2\pi\sqrt{2}}{3}.$$

3.10.5 If the egg has radius R, each slice will have height $2R/n$. A slice can be parametrised by $0 \le \theta \le 2\pi$, $\phi_1 \le \phi \le \phi_2$, with

$$R\cos\phi_1 - R\cos\phi_2 = 2R/n.$$

The area of the part of the surface of the sphere in slice is

$$\int_0^{2\pi}\int_{\phi_1}^{\phi_2} R^2\sin\phi\,d\phi\,d\theta = 2\pi R^2(\cos\phi_1 - \cos\phi_2) = 4\pi R^2/n.$$

This means that each of the n slices has identical area $4\pi R^2/n$.

3.10.6 We project this plane onto the coordinate axes obtaining

$$\int_{\Sigma} xy\,dy\,dz = \int_0^6\int_0^{3-z/2}(3 - y - z/2)y\,dy\,dz = \frac{27}{4},$$

$$-\int_{\Sigma} x^2 \mathrm{d}z\mathrm{d}x = -\int_0^3 \int_0^{6-2x} x^2 \mathrm{d}z\mathrm{d}x = -\frac{27}{2},$$

$$\int_{\Sigma} (x+z)\mathrm{d}x\mathrm{d}y = \int_0^3 \int_0^{3-y} (6-x-2y)\mathrm{d}x\mathrm{d}y = \frac{27}{2},$$

and hence

$$\int_{\Sigma} xy\mathrm{d}y\mathrm{d}z - x^2\mathrm{d}z\mathrm{d}x + (x+z)\mathrm{d}x\mathrm{d}y = \frac{27}{4}.$$

3.11.1 Evaluating this directly would result in evaluating four path integrals, one for each side of the square. We will use Green's Theorem. We have

$$\begin{aligned} \mathrm{d}\omega &= \mathrm{d}(x^3y)\wedge \mathrm{d}x + \mathrm{d}(xy)\wedge \mathrm{d}y \\ &= (3x^2y\mathrm{d}x + x^3\mathrm{d}y)\wedge \mathrm{d}x + (y\mathrm{d}x + x\mathrm{d}y)\wedge \mathrm{d}y \\ &= (y - x^3)\mathrm{d}x\wedge \mathrm{d}y. \end{aligned}$$

The region M is the area enclosed by the square. The integral equals

$$\begin{aligned} \oint_C x^3y\mathrm{d}x + xy\mathrm{d}y &= \int_0^2\int_0^2 (y-x^3)\mathrm{d}x\mathrm{d}y \\ &= -4. \end{aligned}$$

3.11.2 We have

❶ L_{AB} is $y = x$; L_{AC} is $y = -x$, and L_{BC} is clearly $y = -\frac{1}{3}x + \frac{4}{3}$.

❷ We have

$$\begin{aligned} \int_{AB} y^2\mathrm{d}x + x\mathrm{d}y &= \int_0^1 (x^2+x)\mathrm{d}x &&= \frac{5}{6} \\ \int_{BC} y^2\mathrm{d}x + x\mathrm{d}y &= \int_1^{-2} \left(\left(-\frac{1}{3}x+\frac{4}{3}\right)^2 - \frac{1}{3}x\right)\mathrm{d}x &&= -\frac{15}{2} \\ \int_{CA} y^2\mathrm{d}x + x\mathrm{d}y &= \int_{-2}^0 (x^2-x)\mathrm{d}x &&= \frac{14}{3} \end{aligned}$$

Adding these integrals we find

$$\oint_{\triangle} y^2\mathrm{d}x + x\mathrm{d}y = -2.$$

❸ We have

$$\begin{aligned} \int_{\mathscr{D}} (1-2y)\mathrm{d}x\wedge\mathrm{d}y &= \int_{-2}^0 \left(\int_{-x}^{-x/3+4/3} (1-2y)\mathrm{d}y\right)\mathrm{d}x \\ &\quad + \int_0^1 \left(\int_x^{-x/3+4/3} (1-2y)\mathrm{d}y\right)\mathrm{d}x \\ &= -\frac{44}{27} - \frac{10}{27} \\ &= -2. \end{aligned}$$

3.11.6 Observe that

$$\mathrm{d}(x^2+2y^3)\wedge\mathrm{d}y = 2x\mathrm{d}x\wedge\mathrm{d}y.$$

Hence by the generalised Stokes' Theorem the integral equals

$$\int_{\{(x-2)^2+y^2\leq 4\}} 2x\mathrm{d}x\wedge\mathrm{d}y = \int_{-\pi/2}^{\pi/2}\int_0^{4\cos\theta} 2\rho^2\cos\theta\mathrm{d}\rho\wedge\mathrm{d}\theta = 16\pi.$$

To do it directly, put $x - 2 = 2\cos t, y = 2\sin t, 0 \leq t \leq 2\pi$. Then the integral becomes

$$\begin{aligned} \int_0^{2\pi} ((2+2\cos t)^2 + 16\sin^3 t)\mathrm{d}2\sin t &= \int_0^{2\pi} (8\cos t + 16\cos^2 t \\ &\qquad +8\cos^3 t + 32\cos t\sin^3 t)\mathrm{d}t \\ &= 16\pi. \end{aligned}$$

3.11.7 At the intersection path

$$0 = x^2 + y^2 + z^2 - 2(x + y) = (2 - y)^2 + y^2 + z^2 - 4 = 2y^2 - 4y + z^2 = 2(y - 1)^2 + z^2 - 2,$$

which describes an ellipse on the yz-plane. Similarly we get $2(x - 1)^2 + z^2 = 2$ on the xz-plane. We have

$$\mathrm{d}\,(y\mathrm{d}x + z\mathrm{d}y + x\mathrm{d}z) = \mathrm{d}y \wedge \mathrm{d}x + \mathrm{d}z \wedge \mathrm{d}y + \mathrm{d}x \wedge \mathrm{d}z = -\mathrm{d}x \wedge \mathrm{d}y - \mathrm{d}y \wedge \mathrm{d}z - \mathrm{d}z \wedge \mathrm{d}x.$$

Since $\mathrm{d}x \wedge \mathrm{d}y = 0$, by Stokes' Theorem the integral sought is

$$-\int\limits_{2(y-1)^2+z^2\leq 2} \mathrm{d}y\mathrm{d}z - \int\limits_{2(x-1)^2+z^2\leq 2} \mathrm{d}z\mathrm{d}x = -2\pi(\sqrt{2}).$$

(To evaluate the integrals you may resort to the fact that the area of the elliptical region $\dfrac{(x - x_0)^2}{a^2} + \dfrac{(y - y_0)^2}{b^2} \leq 1$ is πab).

If we were to evaluate this integral directly, we would set

$$y = 1 + \cos\theta,\ z = \sqrt{2}\sin\theta, x = 2 - y = 1 - \cos\theta.$$

The integral becomes

$$\int_0^{2\pi} (1 + \cos\theta)\mathrm{d}(1 - \cos\theta) + \sqrt{2}\sin\theta\mathrm{d}(1 + \cos\theta) + (1 - \cos\theta)\mathrm{d}(\sqrt{2}\sin\theta)$$

which in turn

$$= \int_0^{2\pi} \sin\theta + \sin\theta\cos\theta - \sqrt{2} + \sqrt{2}\cos\theta\mathrm{d}\theta = -2\pi\sqrt{2}.$$

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